pion-nucleon 600-Mev scattering resonance and hence occurs in the $I=\frac{1}{2}$ state. Since σ_1' represents only the partial cross section for magnetic dipole photons leading to $j=\frac{3}{2}$, $I=\frac{3}{2}$ states, it is certainly reasonable to assume that $\sigma_1'<100$ μb. Hence we assume $W_1/|V_{12}|^2<2$. In order to estimate R_2, we note that at 450 Mev lab energy the total inelastic cross section for an incident π^+-proton state ($I=\frac{3}{2}$ state) has been measured to be less than 2.5 millibarns.[8] Hence the partial cross section σ_2^{in} must also be less than 2.5 mb. Since the maximum possible partial inelastic cross section σ_2^{max} at this energy is 17.5 mb, it is seen that $R_2<0.15$. By definition W_2 satisfies the inequality $W_2<R_2$. We conclude that R_1, R_2, W_2, and $W_1/|V_{12}|^2$ satisfy the relations $R_1\approx 0$, $R_2<0.15$, $W_2<R_2$, and $W_1/|V_{12}|^2<2$. From these relations and Eq. (2) it can be shown that $\cos(2\theta_{12})>0.84$. Therefore, the phase ϕ_{12} of the amplitude T_{12} must satisfy one of the two inequalities,

$$|\phi_{12}-\delta_2|<17^\circ \quad \text{or} \quad |\phi_{12}-\delta_2-\pi|<17^\circ, \tag{3}$$

where the small photon scattering phase δ_1 has been neglected. Thus the phase relation characteristic of the two-channel case is nearly satisfied here, even though several channels are important.

The amplitudes for production of a p-π^0 state are linear combinations of the amplitudes for production of $I=\frac{1}{2}$ and $I=\frac{3}{2}$ pion-nucleon states. In the model of references 4 and 5 however, the $I=\frac{1}{2}$, $j=\frac{3}{2}$ magnetic dipole state is neglected, so that the phase ϕ_{12} of Eq. (3) above is equal to the phase of the $j=\frac{3}{2}$, magnetic dipole amplitude for the process $\gamma+p\to p+\pi^0$. Similar results may be obtained for the phases of other angular momentum and parity states. In this manner polarization and angular distribution measurements of photoproduction may be related to similar measurements of pion-nucleon scattering.

An interesting discussion concerning this subject was had with Ronald F. Peierls.

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[2] F. Coester, Phys. Rev. 89, 619, (1953).

[3] The choice is definite except that the phase of any state (and hence of all nondiagonal elements of T involving this state) may be increased by π without destroying the symmetry of T. See reference 2.

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[6] P. C. Stein (to be published). Other experimental references are listed in reference 4.

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RADIATIVE CORRECTIONS TO π-e DECAY*

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As is well known, the ratio of probabilities for π-e and π-μ decays is given by[1]

$$R_0=\left(\frac{m_e}{m_\mu}\right)^2\left(\frac{m_\pi^{\,2}-m_e^{\,2}}{m_\pi^{\,2}-m_\mu^{\,2}}\right)^2=1.28\times10^{-4}, \tag{1}$$

neglecting electromagnetic corrections, if the decay interaction is assumed to be

$$ig\bar{\psi}_l\gamma_\mu a\psi_\nu(\partial\varphi_\pi/\partial x_\mu), \tag{2}$$

where $a=(1+i\gamma_5)/2$ and l represents either muon or electron. This interaction is consistent[2] with the hypothesis of universal V-A interaction of Fermi couplings.[3] Recent experiments[4] support these assumptions strongly. The new measurements are becoming sufficiently accurate to justify a calculation of the effect of radiative corrections. This problem has recently been studied by Berman,[5] who has found surprisingly large corrections to π-e decay. In this note it is attempted to understand the reason why the radiative corrections are so large. We are also interested to see whether the pion decay agrees with the recently conjectured "theorem"[6] that the radiative correction to the total probability of a decay process is finite in the limit where the mass of the secondary electron is assumed to be arbitrarily small, although corrections to partial probabilities may be divergent in such a limit.

We first calculated the probability for the inner bremsstrahlung (IB) process in which the pion disintegrates into electron, neutrino, and photon.[7] We integrate it over all neutrino and photon momenta to find the energy spectrum of the electron.[8] Integrating it further over the electron energy, the probability of observing any electron whose energy is less that $E_{max} - \Delta E$ is found to be

$$\frac{\Delta P_{IB}(\Delta E)}{P_0} = \frac{\alpha}{\pi}\left\{-b(\mu)\left[\ln\left(\frac{m_\pi}{2\Delta E}\right)+2\ln(1-\mu^2)-\tfrac{3}{4}\right] - \frac{\mu^2(10-7\mu^2)}{2(1-\mu^2)^2}\ln\mu + \frac{2(1+\mu^2)}{1-\mu^2}L(1-\mu^2) + \frac{15-21\mu^2}{8(1-\mu^2)}\right\}, \quad (3)$$

where P_0 is the uncorrected rate of decay,

$$L(x) = \int_0^x \ln(1-t)(dt/t), \quad \mu = m_e/m_\pi,$$

$$b(\mu) = 2\left(\frac{1+\mu^2}{1-\mu^2}\ln\mu + 1\right),$$

and ΔE is assumed to be small compared with the maximum energy $E_{max} = m_\pi(1+\mu^2)/2$. The total probability of inner bremsstrahlung is given by

$$\frac{\Delta P_{IB}}{P_0} = \frac{\alpha}{\pi}\left\{b(\mu)\left[\ln\left(\frac{\lambda_{min}}{m_\pi}\right) - \ln(1-\mu^2) - \tfrac{1}{2}\ln\mu + \tfrac{3}{4}\right] - \frac{\mu^2(10-7\mu^2)}{2(1-\mu^2)^2}\ln\mu + \frac{2(1+\mu^2)}{1-\mu^2}L(1-\mu^2) + \frac{15-21\mu^2}{8(1-\mu^2)}\right\}. \quad (4)$$

This contains an infrared divergence since photons of very low energy (with infinitesimal mass λ_{min}) are emitted near the maximum electron energy.

To find the correction due to virtual emission and reabsorption of photons, let us note that the interaction (2), or more precisely

$$g\bar{\psi}_l\gamma_\mu a\psi_\nu(i\partial\varphi_\pi/\partial x_\mu - eA_\mu\varphi_\pi), \quad (5)$$

is equivalent to

$$gm_l^0\bar{\psi}_l a\psi_\nu\varphi_\pi, \quad (6)$$

in the lowest order in g and to any order in e, where m_l^0 is the bare mass of the lepton.[9,10] Thus the effect of virtual photons may be regarded as consisting of two parts: (A) the correction to the operator $\bar{\psi}_l a\psi_\nu\varphi_\pi$ due to the dynamical effects of virtual emission of photons, and (B) the correction to the coefficient gm_l^0 that arises when one tries to express it in terms of the observed mass m_l.

The correction A is found to be

$$\frac{\Delta P_A}{P_0} = \frac{\alpha}{\pi}\left[\tfrac{3}{2}\ln\frac{\lambda}{m_\pi} - b(\mu)\left(\ln\frac{\lambda_{min}}{m_\pi} - \tfrac{1}{2}\ln\mu + \tfrac{3}{4}\right) + \frac{\mu^2}{1-\mu^2}\ln\mu + \tfrac{1}{2}\right], \quad (7)$$

where λ is the ultraviolet cutoff. Thus, if the correction B is disregarded for the moment, the corrected rate of π-e decay is given by

$$P = P_0(1+\eta), \quad (8)$$

where

$$\eta = \frac{\alpha}{\pi}\left[\tfrac{3}{2}\ln\left(\frac{\lambda}{m_\pi}\right) - b(\mu)\ln(1-\mu^2) - \frac{\mu^2(8-5\mu^2)}{2(1-\mu^2)^2}\ln\mu + \frac{2(1+\mu^2)}{1-\mu^2}L(1-\mu^2) + \frac{19-25\mu^2}{8(1-\mu^2)}\right], \quad (9)$$

from (4) to (7). As a matter of fact, it is not necessary to calculate the correction B since it is already included in (8) if one remembers that the factor $(m_l^0)^2$ which appears in P_0 is (bare mass)^2 even when the radiative correction is taken into account.

The total decay rate P depends logarithmically on the cutoff λ, but the ratio R of total decay rates for e and μ is independent of the cutoff if it is taken to be the same for both.[11] Under this assumption, the radiative corrections to R_0 of (1) can be expressed as

$$R = R_0(1+\delta)(1+\epsilon), \quad (10)$$

where $1+\delta = [(m_e^0/m_e)/(m_\mu^0/m_\mu)]^2$ comes from correction B and $1+\epsilon$ from correction A and inner bremsstrahlung. Using the electromagnetic mass of order α, one obtains

$$\delta = -(3\alpha/\pi)\ln(m_\mu/m_e) = -16.0(\alpha/\pi). \quad (11)$$

It is seen from (9) that $\epsilon = -0.92(\alpha/\pi)$ to order α.

Note that there is no physical distinction between correction A and correction B, their only role being to modify the decay coupling constant. It is therefore misleading to talk of them separately. As is seen from the smallness of ϵ, however, the sum of correction A and inner bremsstrahlung is quite insensitive to whether the decay particle is muon or electron. Thus the radiative correction to R_0 is mostly due to the correction B. In this sense one might say that, when one measures R, one is actually observing a finite difference of electron and muon self-energies.

Ordinarily the experiment looking for π-e decay excludes electrons whose energy is too low, in order to distinguish them from those of π-μ-e decays. Thus we should like to have the ratio $R(\Delta E)$ = (number of π-e decays for which the energy of the electron is within ΔE of the maximum energy)/(number of π-μ decays). This is obtained from (3) and (8). As is easily seen, $R(\Delta E)$ can be written in the same form as (10) where δ is still given by (11) but ϵ now depends on ΔE. The result agrees with Eq. (2) of reference 5.

Numerically, the radiative correction to the ratio R_0 is -3.9% if all decay electrons are counted. Of this value, -3.7% is due to the mass correction (11). Only -0.2% comes from the inner bremsstrahlung and the virtual photon correction A. If only those electrons are observed whose energy is larger than $E_{\max} - \Delta E$, the correction is -7.8% for $\Delta E \sim 10m_e$ and -14% for $\Delta E \sim 0.5m_e$. Of these, -3.7% are always due to the mass correction δ. The large negative corrections that still remain are nearly equal in magnitude to the positive probabilities of inner bremsstrahlung (3) which are 3.9% for $\Delta E \sim 10m_e$ and 10% for $\Delta E \sim 0.5m_e$. This may be understood qualitatively if one imagines that the probability of finding high-energy electrons is reduced simply because some of them have been shifted to the low-energy side of the spectrum by inner bremsstrahlung. Thus, putting aside the bare mass correction which is energy independent, the large radiative correction found by Berman in $R(\Delta E)$ may be regarded as a consequence of the high efficiency of the π-e system as an emitter of hard photons.

The results of this paper can be qualitatively understood by making use of the "theorem" mentioned at the beginning.[6] At first sight, the radiative correction to π-e decay, although it does not contradict the "theorem," supports it only in a trivial fashion, since not only the total rate but any partial rate of π-e decay vanishes for $m_e \to 0$ because the interaction (5) is proportional to m_e. The fact that the correction δ of (11) diverges logarithmically for $m_e \to 0$ gives no trouble since R_0 tends to zero at the same time. This is in fact expected from our "theorem." It is interesting to note however that the correction ϵ does not diverge in this limit. This can be easily explained by our "theorem," too. For this purpose, we have only to point out that ϵ would be the total radiative correction for pion decay if the interaction were given not by (5) but by

$$f\bar{\psi}_l a\psi_\nu \varphi_\pi, \tag{12}$$

where f is a coupling constant independent of m_l. Since P_0 is now finite for $m_e \to 0$, the correction ϵ cannot afford to diverge if the "theorem" should hold. But (9) is in fact finite for $m_e \to 0$ whereas (3), (4), and (7) diverge. Thus the pion decay gives additional support for the general validity of this conjectured "theorem."

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[8]If the electron mass is neglected in comparison with its energy whenever this approximation does not lead to spurious divergences, the differential spectrum of

electrons is given by

$$dP/P_0 = (\alpha/\pi)(1-x)^{-1}dx\,[(1+x^2)\ln(x/\mu) - 2x - \tfrac{1}{2}(1-x)^2\ln(1-x)],$$

where x is the electron energy measured in units E_{max}. If this is integrated over the range $(0,x)$, one obtains

$$\Delta P_{IB}(x)/P_0 = (\alpha/\pi)\{[2\ln(1-x) + x + \tfrac{1}{2}x^2]\ln(\mu/x) + (\tfrac{9}{4} - \tfrac{1}{2}x + \tfrac{1}{4}x^2)\ln(1-x) + 2L(x) + \tfrac{13}{4}x + \tfrac{1}{8}x^2\}.$$

These formulas are good approximations for any x except at $x \sim 0$. Equation (3), on the other hand, holds only for $\Delta E \ll E_{max}$, where ΔE is related to x by $x = 1 - (2\Delta E/m_\pi)$. Note, however, that no approximation about the electron mass is made in (3), (4), (7), and (9). Thus radiative corrections for π-μ decay are obtained by substituting the observed ratio of pion and muon masses in these formulas.

[9]M. A. Ruderman and W. K. R. Watson, Bull. Am. Phys. Soc. Ser. II, 1, 383 (1956); R. Gatto and M. A. Ruderman, Nuovo cimento 8, 775 (1958).

[10]This is easily proved by rewriting the total Lagrangian making use of the Heisenberg equation of motion for $\bar{\psi}_l$. Obviously (5) and (6) are also equivalent for the inner bremsstrahlung.

[11]If the cutoffs were chosen differently for π-e and π-μ decays, our results concerning virtual photons would become dependent on their ratio ρ. It is interesting to note that measurement of R or $R(\Delta E)$ would determine ρ experimentally if all other assumptions made here were justified.