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$\pi/K \to e\bar{\nu}_e$ branching ratios to $O(e^2p^4)$ in Chiral Perturbation Theory

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Abstract

We calculate the ratios $R_{e/\mu}^{(P)} \equiv \Gamma(P \to e\bar{\nu}_e[\gamma])/\Gamma(P \to \mu\bar{\nu}_\mu[\gamma])$ $(P = \pi, K)$ in Chiral Perturbation Theory to order e^2p^4. We complement the one- and two-loop effective theory results with a matching calculation of the local counterterm, performed within the large-N_C expansion. We find $R_{e/\mu}^{(\pi)} = (1.2352 \pm 0.0001) \times 10^{-4}$ and $R_{e/\mu}^{(K)} = (2.477 \pm 0.001) \times 10^{-5}$, with uncertainty induced by the matching procedure and chiral power counting. Given the sensitivity of upcoming new measurements, our results provide a clean baseline to detect or constrain effects from weak-scale new physics in these rare decays. As a by-product, we also update the theoretical analysis of the individual $\pi(K) \to \ell\bar{\nu}_\ell$ modes.

1 Introduction

The ratio $R_{e/\mu}^{(P)} \equiv \Gamma(P \to e\bar{\nu}_e[\gamma])/\Gamma(P \to \mu\bar{\nu}_\mu[\gamma])$ $(P = \pi, K)$ of leptonic decay rates of light pseudoscalar mesons is helicity-suppressed in the Standard Model (SM), due to the $V - A$ structure of charged current couplings. It is therefore a sensitive probe of all SM extensions that induce pseudoscalar currents and non-universal corrections to the lepton couplings [1]. Recently, attention to these process has been payed in the context of the Minimal Supersymmetric Standard Model, with [2] and without [3] lepton-flavor-violating effects. In general, effects from weak-scale new physics are expected in the range $(\Delta R_{e/\mu})/R_{e/\mu} \sim 10^{-4} - 10^{-2}$ and there is a realistic chance to detect or constrain them because of the following circumstances. (i) First, ongoing experimental searches plan to reach a fractional uncertainty of $(\Delta R_{e/\mu}^{(\pi)})/R_{e/\mu}^{(\pi)} \lesssim 5 \times 10^{-4}$ [4] and $(\Delta R_{e/\mu}^{(K)})/R_{e/\mu}^{(K)} \lesssim 3 \times 10^{-3}$ [5], which represent respectively a factor of 5 and 10 improvement over current errors [6]. (ii) At the same time, the SM theoretical uncertainty can be pushed below this level, since to a first approximation the strong interaction dynamics cancels out in the ratio $R_{e/\mu}$ and hadronic structure dependence appears only through electroweak corrections. Indeed, the most recent theoretical predictions read $R_{e/\mu}^{(\pi)} = (1.2352 \pm 0.0005) \times 10^{-4}$ [7], $R_{e/\mu}^{(\pi)} = (1.2354 \pm 0.0002) \times 10^{-4}$ [8], and $R_{e/\mu}^{(K)} = (2.472 \pm 0.001) \times 10^{-5}$ [8]. In Ref. [7] a general parameterization of the hadronic effects is given, with an estimate of the leading model-independent contributions based on current algebra [9]. The dominant hadronic uncertainty is roughly estimated via dimensional analysis. In Ref. [8], on the other hand, the hadronic component is calculated by modeling the low- and intermediate-momentum region of the loops involving virtual photons.

The primary goal of this investigation is to improve the current status of the hadronic structure dependent effects. To this end, we have analyzed $R_{e/\mu}$ within Chiral Perturbation Theory (ChPT) [10], the low-energy effective field theory (EFT) of QCD. The key feature of this framework is that it provides a controlled expansion of the amplitudes in terms of the masses of pseudoscalar mesons and charged leptons ($p \sim m_{\pi,K,\ell}/\Lambda_\chi$, with $\Lambda_\chi \sim 4\pi F_\pi \sim 1.2\,\mathrm{GeV}$), and the electromagnetic coupling (e). Electromagnetic corrections to (semi)-leptonic decays of K and π have been worked out to $O(e^2p^2)$ [11, 12], but had never been pushed to $O(e^2p^4)$, as required for $R_{e/\mu}$ in order to match the experimental accuracy. In this work we report full details of our analysis of $R_{e/\mu}$ to $O(e^2p^4)$, while a summary of the results is presented elsewhere [13]. To the order we work in ChPT, $R_{e/\mu}$ features both model independent double chiral logarithms (previously neglected) and an a priori unknown low-energy coupling (LEC). By including the finite loop effects and estimating the LEC via a matching calculation in large-N_C QCD, we thus provide the first complete result of $R_{e/\mu}$ to $O(e^2p^4)$ in the EFT power counting. Most importantly, the matching calculation allows us to further reduce the theoretical uncertainty and put it on more solid ground.

Our presentation is organized as follows. In Section 2 we introduce the basic definitions and outline the strategy to calculate $R_{e/\mu}$ to $O(e^2p^4)$. In Section 3 we shortly review the basic ChPT formalism and the needed effective lagrangians. The loop calculation is described in Section 4 and in Appendix A, and the results are reported in Section 5. We then report the matching calculation of the effective coupling in Section 6, with technical details in Appendix B. We present the contribution from real photon emission in Section 7, while in Section 8 we give our final analytical and numerical results for $R_{e/\mu}^{(\pi,K)}$ and discuss them. Section 9 is devoted to updating the theoretical expression for the individual $\pi(K) \to \ell\bar{\nu}_\ell$ rates. Finally, Section 10 contains our concluding remarks. Since we are reporting here the first ChPT calculation to order e^2p^4, we give several details and intermediate steps of our analysis, both throughout the text and in the Appendixes.

2 $R_{e/\mu}^{(\pi,K)}$ in ChPT: overview

To avoid excessive notational clutter, throughout this paper we illustrate the main arguments in the case of $\pi \to \ell\nu$ decays and subsequently report any significant changes that occur for K decays. We consider the ratio

$$R_{e/\mu}^{(\pi)} = \frac{\Gamma\left(\pi^+ \to e^+\nu_e(\gamma)\right)}{\Gamma\left(\pi^+ \to \mu^+\nu_\mu(\gamma)\right)} \tag{1}$$

to order e^2p^4 in Chiral Perturbation Theory (ChPT). Within ChPT the invariant amplitudes [1] can be expanded in powers of the external masses and momenta (of both pseudoscalar mesons and leptons) and powers of the electromagnetic coupling. To leading order in the chiral expansion one finds

$$T_\ell^{p^2} = -i2G_F V_{ud}^* F\, m_\ell\, \bar{u}_L(p_\nu)\, v(p_\ell) \ . \tag{2}$$

F can be identified to lowest order with F_π (and F_K, F_η). Setting $e = 0$, to a given order (p^{2n}) in the purely "strong" chiral expansion, the amplitude reads as above, with the replacement $F \to F_\pi^{(2n)}$, $F_\pi^{(2n)}$ being the pion decay constant to order p^{2n}. When considering the ratio of electron-to-muon decay rates the pion decay constant drops and one obtains the well known expression:

$$R_{e/\mu}^{(0),(\pi)} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 \ . \tag{3}$$

Non-trivial corrections to Eq. 3 arise only when $e \neq 0$, i.e. to order e^2p^{2n} in ChPT.

Lorentz invariance implies that higher order contributions are proportional to the lowest order amplitude, and this allows one to write to $O(e^2p^4)$

$$\Gamma(\pi \to \ell\nu[\gamma]) = \Gamma^{(0)}(\pi \to \ell\nu) \times \left[1 + 2\,\mathrm{Re}\left(r_\ell^{e^2p^2} + r_\ell^{e^2p^4}\right) + \delta_\ell^{e^2p^2} + \delta_\ell^{e^2p^4}\right] \ , \tag{4}$$

[1] Intermediate steps in our analysis depend on the definition of the invariant amplitude T_ℓ ($\ell = \mu, e$), for which we use ${}_{\mathrm{out}}\langle \ell^+(p_\ell)\nu_\ell(p_\nu)|\pi^+(p)\rangle_{\mathrm{in}} = (2\pi)^4\delta^{(4)}\left(p - p_\ell - p_\nu\right)\ i\,T_\ell$.

where

$$\Gamma^{(0)}(\pi \to \ell\nu) = \frac{G_F^2 |V_{ud}|^2 F_\pi^2}{4\pi} \, m_\pi \, m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2 \tag{5}$$

and

$$r_\ell^{e^2p^{2n}} = \frac{T_\ell^{e^2p^{2n}}}{T_\ell^{p^2}} \tag{6}$$

$$\delta_\ell^{e^2p^{2n}} = \frac{\Gamma(\pi \to \ell\nu\gamma)|_{e^2p^{2n}}}{\Gamma^{(0)}(\pi \to \ell\nu)} \tag{7}$$

are respectively the corrections induced by virtual and real photon effects, whose sum is free of infrared divergences. Taking the ratio of electron and muon decay rates one obtains:

$$R_{e/\mu}^{(\pi)} = R_{e/\mu}^{(0),(\pi)} \left[1 + \Delta_{e^2p^2}^{(\pi)} + \Delta_{e^2p^4}^{(\pi)} + ...\right] \tag{8}$$

$$\Delta_{e^2p^{2n}}^{(\pi)} = 2\,\mathrm{Re}\left(r_e^{e^2p^{2n}} - r_\mu^{e^2p^{2n}}\right) + \left(\delta_e^{e^2p^{2n}} - \delta_\mu^{e^2p^{2n}}\right) \tag{9}$$

The main feature emerging from Eq. 9 is that only those diagrams that depend in a non-trivial way on the lepton mass contribute to $R_{e/\mu}$. The diagrams leading to m_ℓ-independent $r_\ell^{e^2p^{2n}}$ will drop when taking the difference of electron and muon amplitudes. This observation greatly reduces the number of diagrams to be calculated in the effective theory. All the considerations presented in this section trivially extend to the case of leptonic decays of charged kaons ($K \to \ell\nu$).

3 Electromagnetic corrections to (semi)-leptonic processes at low energy

The appropriate theoretical framework for the analysis of electromagnetic effects in semileptonic kaon decays is a low-energy effective quantum field theory where the asymptotic states consist of the pseudoscalar octet, the photon and the light leptons [11]. The corresponding lowest-order effective Lagrangian is given by

$$\mathcal{L}_{\rm eff} = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle + e^2 F^4 Z \langle \mathcal{Q}_{\rm L}^{\rm em} \mathcal{Q}_{\rm R}^{\rm em} \rangle - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_\ell [\bar{\ell}(i\,\partial\!\!\!/ + e A\!\!\!/ - m_\ell)\ell + \overline{\nu_{\ell {\rm L}}}\, i\,\partial\!\!\!/ \nu_{\ell {\rm L}}]. \tag{10}$$

F denotes the pion decay constant in the chiral limit and in the absence of electroweak interactions. The low energy constant $Z \simeq 0.8$ can be determined by mass splitting of charged and neutral pions. The symbol $\langle \; \rangle$ denotes the trace in three-dimensional flavour space, and

$$u_\mu = i[u^\dagger(\partial_\mu - i r_\mu)u - u(\partial_\mu - i l_\mu)u^\dagger] \, , \tag{11}$$

with the Goldstone modes collected in the field u:

$$u = \exp\left[\frac{i\Phi}{\sqrt{2}F}\right] \qquad \Phi = \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{bmatrix}. \tag{12}$$

The photon field A_μ and the leptons ℓ, ν_ℓ ($\ell = e, \mu$) are contained in (11) by adding appropriate terms to the usual external vector and axial-vector sources v_μ, a_μ:

$$\begin{aligned} l_\mu &= v_\mu - a_\mu - eQ_{\rm L}^{\rm em} A_\mu + \sum_\ell (\bar{\ell}\gamma_\mu \nu_{\ell {\rm L}} Q_{\rm L}^{\rm w} + \overline{\nu_{\ell {\rm L}}}\gamma_\mu \ell Q_{\rm L}^{\rm w\dagger}), \\ r_\mu &= v_\mu + a_\mu - eQ_{\rm R}^{\rm em} A_\mu. \end{aligned} \tag{13}$$

The 3×3 matrices $Q_{\rm L,R}^{\rm em}$, $Q_{\rm L}^{\rm w}$ are spurion fields. At the end, one identifies $Q_{\rm L,R}^{\rm em}$ with the quark charge matrix

$$Q^{\rm em} = \begin{bmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}, \tag{14}$$

whereas the weak spurion is taken at

$$Q_{\rm L}^{\rm w} = -2\sqrt{2}\, G_{\rm F} \begin{bmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

where $G_{\rm F}$ is the Fermi coupling constant and V_{ud}, V_{us} are Cabibbo-Kobayashi–Maskawa matrix elements. For the construction of the effective Lagrangian it is also convenient to define

$$\mathcal{Q}_{\rm L}^{\rm em,w} := uQ_{\rm L}^{\rm em,w} u^\dagger, \qquad \mathcal{Q}_{\rm R}^{\rm em} := u^\dagger Q_{\rm R}^{\rm em} u. \tag{16}$$

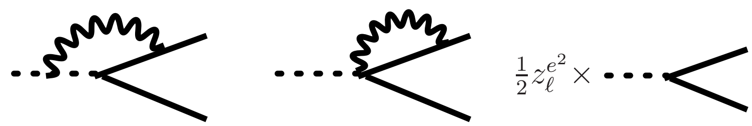
Explicit chiral symmetry breaking is included in $\chi_+ = u^\dagger \chi u^\dagger + u\chi^\dagger u$ where χ is proportional to the quark mass matrix:

$$\chi = 2B_0 \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix}, \tag{17}$$

and the factor B_0 is related to the quark condensate in the chiral limit by $\langle 0|\overline{q}q|0\rangle = -F^2 B_0$.

The local action at next-to-leading order involves the sum of three terms, $\mathcal{L}_{p^4} + \mathcal{L}_{e^2p^2}^{\rm str} + \mathcal{L}_{e^2p^2}^{\rm lept}$. The first one, $\mathcal{L}_{p^4}$ includes the well-known Gasser-Leutwyler Lagrangian [14] in the presence of the generalized external sources introduced in (13), as well as a term from the Wess-Zumino-Witten functional that incorporates the effect of chiral anomalies [15]. Here we quote only the operators relevant to our analysis:

$$\begin{aligned} \mathcal{L}_{p^4} \supset\ & -iL_9\, \langle f_+^{\mu\nu} u_\mu u_\nu \rangle + \frac{L_{10}}{4} \langle f_{+\mu\nu} f_+^{\mu\nu} - f_{-\mu\nu} f_-^{\mu\nu} \rangle \\ & - \frac{iN_C}{48\pi^2} \varepsilon^{\mu\nu\alpha\beta} \langle \Sigma_\mu^L U^\dagger \partial_\nu r_\alpha U \ell_\beta - \Sigma_\mu^R U \partial_\nu \ell_\alpha U^\dagger r_\beta + \Sigma_\mu^L \ell_\nu \partial_\alpha \ell_\beta + \Sigma_\mu^L \partial_\nu \ell_\alpha \ell_\beta \rangle \end{aligned} \tag{18}$$



(a) (b) (c)

Figure 1: Diagrams contributing to $R_{e/\mu}$ to order $e^2 p^2$. Dashed lines indicate pseudoscalar mesons, solid lines leptons, and wavy lines photons.

with

$$
\begin{aligned}
f_{\pm}^{\mu\nu} &= u F_{\mathrm{L}}^{\mu\nu} u^\dagger \pm u^\dagger F_{\mathrm{R}}^{\mu\nu} u, \\
F_{\mathrm{L}}^{\mu\nu} &= \partial^\mu l^\nu - \partial^\nu l^\mu - i[l^\mu, l^\nu], \\
F_{\mathrm{R}}^{\mu\nu} &= \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu], \\
U &= u^2, \\
\Sigma_\mu^L &= U^\dagger \partial_\mu U, \\
\Sigma_\mu^R &= U \partial_\mu U^\dagger.
\end{aligned} \tag{19}
$$

The second term, $\mathcal{L}_{e^2p^2}^{\mathrm{srt}}$, encodes the interaction of ultraviolet (UV) virtual photons with hadronic degrees of freedom [16, 17, 18]. It contributes to the individual $P \to e\nu$ and $P \to \mu\nu$, but leads to an m_ℓ-independent $r_\ell^{e^2p^2}$ so that it has no effect on $R_{e/\mu}$. The same argument applies to $\mathcal{L}_{e^2p^2}^{\mathrm{lept}}$, which involves leptonic bilinears. Similarly, when inserted in one-loop purely mesonic graphs, these effective operators contribute to $P \to e\nu$ and $P \to \mu\nu$ to order e^2p^4, but their contribution cancels in $R_{e/\mu}$. Therefore, there is no need to report the full expression of these effective lagrangians here.

Finally, we shall see that a counterterm of $O(e^2p^4)$ is needed in order to make $R_{e/\mu}$ finite to $O(e^2p^4)$. While we have not constructed the most general $\mathcal{L}_{e^2p^4}^{\mathrm{lept}}$, on the basis of power counting we can conclude that the same combination of operators (and LECs) contributes to both $R_{e/\mu}^{(K)}$ and $R_{e/\mu}^{(\pi)}$. This fact is also explicitly borne out in the matching calculation that we perform in Section 6.

4 Virtual-photon corrections: analysis

We work in Feynman gauge, use dimensional regularization to deal with ultraviolet (UV) divergences and an infinitesimal photon mass to deal with infrared (IR) divergences. We report the diagrams contributing to $R_{e/\mu}$ to $O(e^2p^2)$ and $O(e^2p^4)$ in Fig. 1 and Figs. 2-3, respectively. At the order we work, we need the charged lepton and pseudoscalar meson wavefunction renormalizations to one-loop accuracy. We denote them by $Z_\ell = 1 + z_\ell^{e^2}$ (charged lepton) and $Z_\pi = 1 + z_\pi^{p^2} + z_\pi^{e^2}$ (pseudoscalar meson).

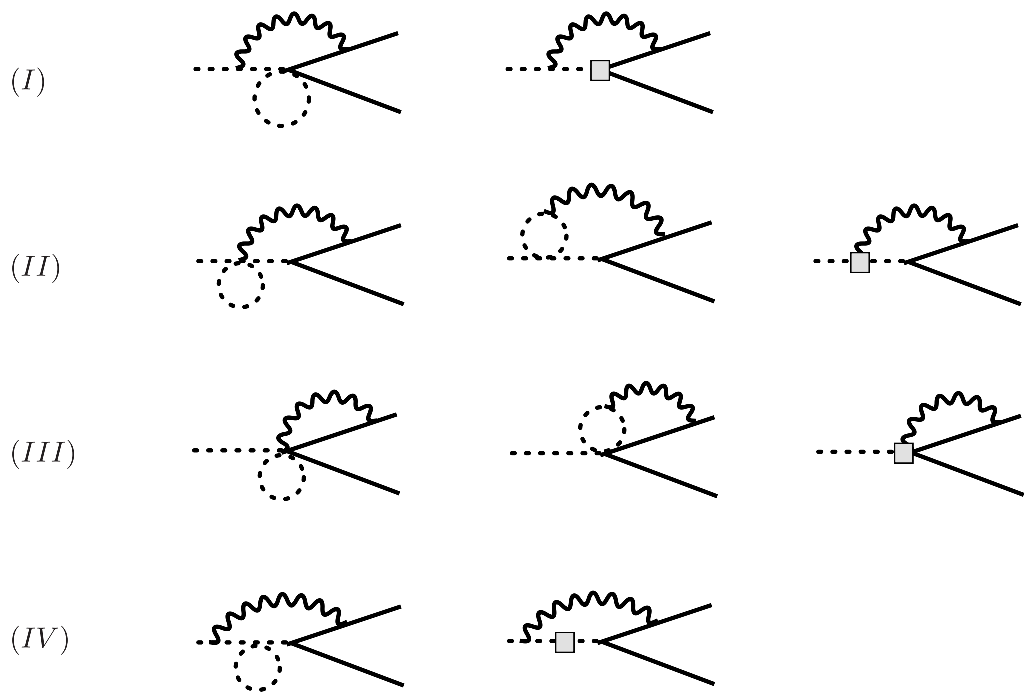


Figure 2: 1PI diagrams contributing to $R_{e/\mu}$ to order e^2p^4. Shaded squares indicate vertices from the $O(p^4)$ effective lagrangian.

To order e^2p^2 one has to consider only two 1PI diagrams and the effect of charged lepton wave-function renormalization (see Fig. 1). The resulting amplitude $T_\ell^{e^2p^2}$ [11] coincides with the point-like approximation of Ref. [19]. Since this is well known, we do not dwell further on it, but we will simply report the result in the next section. The situation is more interesting to next-to-leading order.

4.1 Organizing the $O(e^2p^4)$ diagrams

To $O(e^2p^4)$ one has to consider (i) two-loop graphs with vertices from the lowest order effective lagrangian and (ii) one-loop graphs with one insertion from the NLO lagrangian $\mathcal{L}_{p^4}$ (we denote the latter vertices with shaded squares); (iii) a tree level diagram with insertion of a local operator of $O(e^2p^4)$. In Fig. 2 we report all relevant 1PI topologies: each $O(p^4)$ vertex receives contributions from several $O(p^4)$ operators and all allowed mesons run in the internal loops. External leg corrections are depicted in Fig. 3.

The self-energy insertion on the internal mesonic leg (class (IV) in Fig. 2) is handled by observing that to $O(p^4)$ the self-energy reads $\Sigma(p^2) = A + Bp^2$ (with A and B momentum-

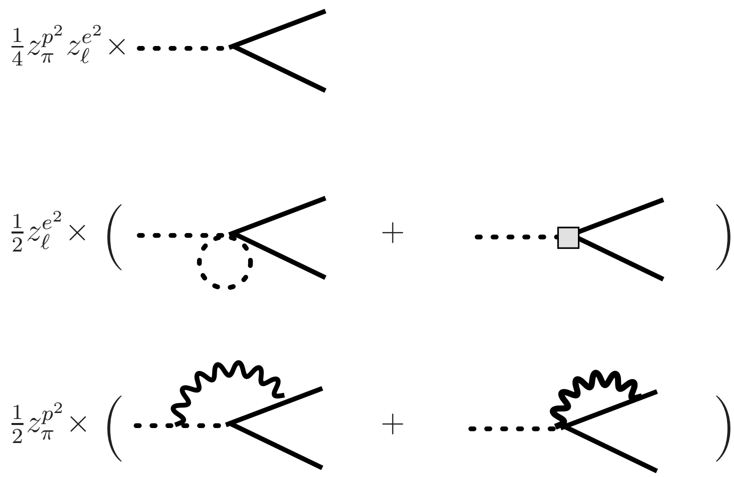


Figure 3: External leg corrections to $R_{e/\mu}$ to order e^2p^4.

independent) and therefore

$$\frac{i}{p^2-m_0^2}\left(-i\Sigma(p^2)\right)\frac{i}{p^2-m_0^2} = (Z-1)\frac{i}{p^2-m^2} + (m^2-m_0^2)\frac{\partial}{\partial m_0^2}\frac{i}{p^2-m_0^2} \ , \tag{20}$$

where Z represents the on-shell wave-function renormalization, m_0 is the $O(p^2)$ mass and m is the physical $O(p^4)$ mass. With this result at hand, by re-grouping the diagrams of class (IV) and external leg corrections with those of classes (I), (II), and (III), it is straightforward to show that the inclusion of virtual corrections to $O(e^2p^4)$ amounts to:

- using the physical $O(p^4)$ meson mass in the amplitude of $O(e^2p^2)$;
- calculating a set of "effective" one-loop diagrams with vertices given by appropriate off-shell form factors evaluated to $O(p^4)$ in d-dimensions. These effective one-loop diagrams are shown in Fig. 4. The shaded circles denote respectively: the d-dimensional $O(p^4)$ $\pi\ell\nu$ vertex (Fig. 4(a) and (d), with off-shell pion and charged lepton in Fig. 4(a)); the d-dimensional $O(p^4)$ $\pi\pi\gamma$ vertex with the photon and one pion off-shell (Fig. 4(b)); the d-dimensional $O(p^4)$ $\pi\ell\nu\gamma$ vertex with the photon and charged lepton off-shell (Fig. 4 (c)).

Within this approach one starts the calculation of genuine two-loop diagrams at a stage where the one-loop sub-divergences (generating non-local singularities) have already been subtracted. As we shall see, another advantage is that the non-local O(p^4) vertices admit a simple dispersive parameterization that greatly simplifies the calculation.

As seen from Fig. 4, the virtual photon contributions can be divided into 1PI and external leg corrections. For the external leg corrections we find:

$$T_\ell^{e^2p^4}\Big|_{\mathrm{non}-1PI} = \frac{1}{2}z_\ell^{e^2}\ \left(\frac{F_\pi^{(4)}}{F}-1\right)\times T_\ell^{p^2}\ , \tag{21}$$

where $F_\pi^{(4)}/F$ has to be evaluated in d-dimensions. The 1PI contribution can be written as the sum of the mass-renormalization in $T_\ell^{e^2p^2}$ and a convolution:

$$\begin{aligned} T_\ell^{e^2p^4}\Big|_{1PI} &= 2G_F V_{ud}^* e^2 F \int \frac{d^dq}{(2\pi)^d} \frac{\bar{u}_L(p_\nu)\gamma^\nu \left[-(\not{p}_\ell - \not{q}) + m_\ell\right]\gamma^\mu v(p_\ell)}{\left[q^2 - 2q\cdot p_\ell + i\epsilon\right]\left[q^2 - m_\gamma^2 + i\epsilon\right]} T_{\mu\nu}^{V-A}(p,q) \\ &+ \left(m_\pi^2\big|_{p^4} - m_\pi^2\big|_{p^2}\right) \frac{\partial}{\partial m_\pi^2} T_\ell^{e^2p^2} , \end{aligned} \tag{22}$$

where

$$T_{\mu\nu}^{V-A} = \frac{1}{\sqrt{2}F} \int dx\, e^{iqx+iWy} \langle 0|T(J_\mu^{EM}(x)\,(V_\nu - A_\nu)(y)|\pi^+(p)\rangle , \tag{23}$$

with $V_\mu(A_\mu) = \bar{u}\gamma_\mu(\gamma_5)d$ and $W = p - q$. Lorentz invariance and Ward identities imply that $T_{\mu\nu}^{V-A}$ in turn can be decomposed as follows (see also [20]): [2]

$$\begin{aligned} \left(T^{V-A}\right)^{\mu\nu}(p,q) &= iV_1\,\epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta + \left[\frac{(2p-q)^\mu(p-q)^\nu}{2p\cdot q - q^2} + g^{\mu\nu}\right]\left(\frac{F_\pi^{(4)}}{F} - 1\right) \\ &- A_1\,(q\cdot p g^{\mu\nu} - p^\mu q^\nu) - (A_2 - A_1)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \\ &+ \left[\frac{(2p-q)^\mu(p-q)^\nu}{2p\cdot q - q^2} - \frac{q^\mu(p-q)^\nu}{q^2}\right]\left(F_V^{\pi\pi}(q^2) - 1\right) \\ &- A_3\left[q\cdot p\,(q^\mu p^\nu - q^\mu q^\nu) + q^2\,(p^\mu q^\nu - p^\mu p^\nu)\right] \end{aligned} \tag{24}$$

The form factors V_1, A_i depend in general on both q^2 and $W^2 = (p-q)^2$ and have to be evaluated to $O(p^4)$ in ChPT in d-dimensions [3]. The same applies to the pion form factor $F_V^{\pi\pi}(q^2)$ and decay constant F_π. The convolution integral generates a term proportional to $T_\ell^{e^2p^2}|_{1PI}$ as well as terms induced by V_1, $A_{1,2}$, and $F_V^{\pi\pi} - 1$. With obvious notation we can write

$$\begin{aligned} T_\ell^{e^2p^4}\Big|_{1PI} &= T_{V_1} + T_{A_1} + T_{A_2} + T_{F_V} \\ &+ \left(\frac{F_\pi^{(4)}}{F} - 1\right) T_\ell^{e^2p^2}\Big|_{1PI} + \left(m_\pi^2\big|_{p^4} - m_\pi^2\big|_{p^2}\right) \frac{\partial}{\partial m_\pi^2} T_\ell^{e^2p^2} . \end{aligned} \tag{25}$$

Combining Eqs. 25 and 21 we then obtain:

$$T_\ell^{e^2p^4} = T_{V_1} + T_{A_1} + T_{A_2} + T_{F_V} + \left(\frac{F_\pi^{(4)}}{F} - 1\right) T_\ell^{e^2p^2} + \left(m_\pi^2\big|_{p^4} - m_\pi^2\big|_{p^2}\right) \frac{\partial}{\partial m_\pi^2} T_\ell^{e^2p^2} \tag{26}$$

The effect of the the last two terms in Eq. 26 is taken into account by simply using the physical pion mass and decay constant to $O(p^4)$ in $T_\ell^{e^2p^2}$. The remaining terms provide a genuine shift to the invariant amplitude. In order to calculate such a shift, we need to:

[2] In this work we use the convention $\epsilon_{0123} = +1$ for the Levi Civita symbol.

[3] To $O(p^4)$ the form factor A_3 vanishes.

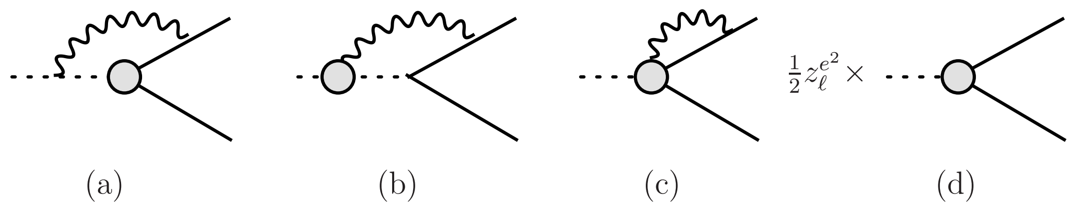


Figure 4: Effective one-loop diagrams contributing to $R_{e/\mu}$ to order $e^2 p^4$. The shaded circles represent the $O(p^4)$ contribution to d-dimensional off-shell effective vertices.

(i) Work out the relevant form factors $V_1, A_{1,2}, F_V^{\pi\pi}$ to $O(p^4)$ (one-loop) in d-dimensions.

(ii) Insert them in the convolution representation of Eq. 22 and calculate the resulting integrals.

In the following subsections we report the results of these steps.

4.2 Form factors in d-dimensions

We work in d dimension with $d = 4 + 2w$ [21, 22]. The relevant form factors to $O(p^4)$ read:

$$V_1 = -\frac{N_C}{24\pi^2 F^2} \tag{27}$$

$$A_1 = -\frac{4\,(L_9 + L_{10})}{F^2} \tag{28}$$

$$A_2 = -2\frac{(F_V^{\pi\pi}(q^2) - 1)}{q^2} \tag{29}$$

$$F_V^{\pi\pi}(q^2) = 1 + 2\,H_{\pi\pi}(q^2) + H_{KK}(q^2) \tag{30}$$

The loop function $H_{aa}(q^2)$ [14] reads

$$F^2 H_{aa}(q^2) = q^2 \left[\frac{A(m_a^2)}{m_a^2} \frac{d-2}{8(d-1)} + \frac{2}{3} L_9 \right] + \frac{q^2 - 4m_a^2}{4(d-1)} \bar{J}^{aa}(q^2) \ , \tag{31}$$

with

$$A(m^2) \equiv -i \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\epsilon} = -\frac{m^{2+2w}}{(4\pi)^{2+w}} \Gamma(-1-w) \tag{32}$$

$$\bar{J}^{aa}(q^2) = J^{aa}(q^2) - J^{aa}(0) \tag{33}$$

$$\begin{aligned} J^{aa}(q^2) &\equiv -i \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 - m_a^2 + i\epsilon] \ [(k-q)^2 - m_a^2 + i\epsilon]} \\ &= \frac{1}{(4\pi)^{2+w}} \Gamma(-w) \int_0^1 dx \left[m_a^2 - q^2\, x(1-x) \right]^w \end{aligned} \tag{34}$$

The function $J^{aa}(q^2)$ admits a dispersive representation [21, 22] in d-dimensions, which proves very useful in the evaluation of genuine two-loop contributions:

$$J^{aa}(q^2) = m_a^{2w} \int_{4m_a^2}^{\infty} [d\sigma] \, \frac{1}{\sigma - q^2} \tag{35}$$

$$[d\sigma] = \frac{d\sigma}{(4\pi)^{2+w}} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+w\right)} \left(\frac{\sigma}{4m_a^2} - 1\right)^{w} \left(1 - \frac{4m_a^2}{\sigma}\right)^{1/2} \tag{36}$$

4.3 "Effective" one-loop diagrams

The 1PI contributions $T_{V_1}, T_{A_1}, T_{A_2}, T_{F_V}$ can be written as the convolution of a known kernel with the d-dimensional form factors $V_1, A_{1,2}, F_V^{\pi\pi}$. It is simple to check that $T_{V_1}, T_{A_1}, T_{A_2}, T_{F_V}$ are IR finite, so we set $m_\gamma = 0$. Upon inserting the $O(p^4)$ ChPT form factors into Eqs. 24 and 22, we obtain:

$$T_{V_1} = T_\ell^{p^2} \, e^2 V_1 \, \frac{A(m_\ell^2)}{2(d-1)m_\ell^2} \, \left[(4+d)m_\ell^2 - (d-2)m_\pi^2\right] \tag{37}$$

$$T_{A_1} = -T_\ell^{p^2} \, e^2 A_1 \, \frac{A(m_\ell^2)}{4(d-1)m_\ell^2} \, \left[(3d^2 - 6d + 4)m_\ell^2 - (d-2)^2 m_\pi^2\right] \tag{38}$$

$$\begin{aligned} T_{A_2} = & -2\, T_\ell^{p^2} \, e^2 \Bigg\{ d\ (2a_{\pi\pi} + a_{KK})\, A(m_\ell^2) \\ & + \frac{2b_{\pi\pi}}{i} \left[2I_2^{(\ell)\pi\pi} + I_2^{(\ell)KK} - 8m_\pi^2 I_1^{(\ell)\pi\pi} - 4m_K^2 I_1^{(\ell)KK}\right] \\ & + \frac{b_{\pi\pi}}{i m_\ell^2} \left(1 - \frac{d}{2}\right) \Bigg[2\left(I_4^{\pi\pi} - I_5^{\pi\pi}\right) + \left(I_4^{KK} - I_5^{KK}\right) \\ & - 8m_\pi^2 \left(I_3^{\pi\pi} - I_2^{(\ell)\pi\pi}\right) - 4m_K^2 \left(I_3^{KK} - I_2^{(\ell)KK}\right) \Bigg] \Bigg\} \end{aligned} \tag{39}$$

$$\begin{aligned} T_{F_V} = & \ 2\, T_\ell^{p^2} \, e^2 \Bigg\{ (2a_{\pi\pi} + a_{KK}) \ \frac{m_\pi^2 A(m_\pi^2) - m_\ell^2 A(m_\ell^2)}{m_\pi^2 - m_\ell^2} \\ & + \frac{m_\pi^2}{m_\pi^2 - m_\ell^2} \, \frac{b_{\pi\pi}}{i} \left(2I_2^{(\ell)\pi\pi} + I_2^{(\ell)KK} - 8m_\pi^2 I_1^{(\ell)\pi\pi} - 4m_K^2 I_1^{(\ell)KK}\right) \\ & - \frac{m_\ell^2}{m_\pi^2 - m_\ell^2} \frac{b_{\pi\pi}}{i} \left(2I_2^{(\pi)\pi\pi} + I_2^{(\pi)KK} - 8m_\pi^2 I_1^{(\pi)\pi\pi} - 4m_K^2 I_1^{(\pi)KK}\right) \\ & + (m_\pi^2 + m_\ell^2) \frac{b_{\pi\pi}}{i} \left(2T_2^{\pi\pi} + T_2^{KK} - 8m_\pi^2 T_1^{\pi\pi} - 4m_K^2 T_1^{KK}\right) \Bigg\} \end{aligned} \tag{40}$$

In the above expressions we have used the definitions:

$$a_{\pi\pi} = \frac{1}{F^2}\left(\frac{A(m_\pi^2)}{m_\pi^2}\frac{d-2}{8(d-1)} + \frac{2}{3}L_9\right) \tag{41}$$

$$a_{KK} = \frac{1}{F^2}\left(\frac{A(m_K^2)}{m_K^2}\frac{d-2}{8(d-1)} + \frac{2}{3}L_9\right) \tag{42}$$

$$b_{\pi\pi} = \frac{1}{4(d-1)\,F^2} \tag{43}$$

which come from the decomposition $H_{mm}(q^2) = a_{mm}q^2 + b_{mm}(q^2 - 4m_m^2)\bar{J}^{mm}(q^2)$. Moreover, the building-block two loop integrals are defined as follows:

$$I_1^{(\ell)aa} = \int \frac{d^dq}{(2\pi)^d}\,\frac{\bar{J}^{aa}(q^2)}{q^2\,(q^2 - 2q\cdot p_\ell)} \tag{44}$$

$$I_1^{(\pi)aa} = \int \frac{d^dq}{(2\pi)^d}\,\frac{\bar{J}^{aa}(q^2)}{q^2\,(q^2 - 2q\cdot p)} \tag{45}$$

$$I_2^{(\ell)aa} = \int \frac{d^dq}{(2\pi)^d}\,\frac{\bar{J}^{aa}(q^2)}{q^2 - 2q\cdot p_\ell} \tag{46}$$

$$I_2^{(\pi)aa} = \int \frac{d^dq}{(2\pi)^d}\,\frac{\bar{J}^{aa}(q^2)}{q^2 - 2q\cdot p} \tag{47}$$

$$I_3^{aa} = \int \frac{d^dq}{(2\pi)^d}\,\frac{\bar{J}^{aa}(q^2)}{q^2} \tag{48}$$

$$I_4^{aa} = \int \frac{d^dq}{(2\pi)^d}\,\bar{J}^{aa}(q^2) \tag{49}$$

$$I_5^{aa} = \int \frac{d^dq}{(2\pi)^d}\,\frac{q^2\,\bar{J}^{aa}(q^2)}{q^2 - 2q\cdot p_\ell} \tag{50}$$

$$T_1^{aa} = \int \frac{d^dq}{(2\pi)^d}\,\frac{\bar{J}^{aa}(q^2)}{q^2\,(q^2 - 2q\cdot p_\ell)\,(q^2 - 2q\cdot p)} \tag{51}$$

$$T_2^{aa} = \int \frac{d^dq}{(2\pi)^d}\,\frac{\bar{J}^{aa}(q^2)}{(q^2 - 2q\cdot p_\ell)\,(q^2 - 2q\cdot p)}\;. \tag{52}$$

The evaluation of these integrals can be done analytically and is reported in Appendix A.

4.4 K decays

The procedure outlined above remains true for the analysis of $K \to \ell\nu$. In the convolution kernel one has to simply replace $p \to p_K$ ($p^2 = m_\pi^2 \to p_K^2 = m_K^2$). The form factors V_1 and A_1 remain unchanged, while in A_2 one has to replace $F_V^{\pi\pi}(q^2) \to F_V^{KK}(q^2) = 1 + 2H_{KK}(q^2) + H_{\pi\pi}(q^2)$, which again amounts to the interchange $m_\pi \leftrightarrow m_K$. As a consequence, the full result for $T_\ell^{e^2p^4}(K \to \ell\nu)$ can be obtained from the pion case by interchanging everywhere m_π with m_K.

5 Virtual-photon corrections: results

We collect here the results for $r_\ell^{e^2p^{2n}} = T_\ell^{e^2p^{2n}}/T_\ell^{p^2}$. Since $R_{e/\mu} \propto r_e^{e^2p^{2n}} - r_\mu^{e^2p^{2n}}$, we systematically neglect m_ℓ-independent contributions to $r_\ell^{e^2p^{2n}}$ that would drop in the difference. We also introduce the notation:

$$z_\ell \equiv \left(\frac{m_\ell}{m_\pi}\right)^2 \qquad z_\gamma \equiv \left(\frac{m_\gamma}{m_\pi}\right)^2 \qquad \tilde{z}_\ell \equiv \left(\frac{m_\ell}{m_K}\right)^2 \qquad \tilde{z}_\pi \equiv \left(\frac{m_\pi}{m_K}\right)^2 . \tag{53}$$

5.1 Leading order: $r_\ell^{e^2p^2}$

The one loop virtual photon contributions read [11]:

$$\begin{aligned} r_\ell^{e^2p^2} &= -\frac{\alpha}{2\pi} \log\sqrt{z_\gamma} \left[\frac{1+z_\ell}{1-z_\ell}\log z_\ell\right] \\ &+ \frac{\alpha}{4\pi}\left[\frac{7}{2}\log\frac{m_\ell^2}{\mu^2} + \log z_\ell - \frac{2}{1-z_\ell}\log z_\ell + \frac{1}{2}\frac{1+z_\ell}{1-z_\ell}(\log z_\ell)^2\right] . \end{aligned} \tag{54}$$

Note that the dependence on the renormalization scale μ drops in $R_{e/\mu}$. Moreover, the dependence on the IR regulator m_γ disappears once the effect of real photon emission is included.

5.2 Next to leading order: $r_\ell^{e^2p^4}$

Using the notation $\bar{L}_{9,10} \equiv (4\pi)^2\ L_{9,10}^r(\mu)$ and $\ell_\alpha = \log\frac{m_\alpha^2}{\mu^2}$ with $\alpha = \pi, K, \ell$, we find the following expressions for the divergent and finite parts of the $O(e^2p^4)$ amplitudes:

$$r_\ell^{e^2p^4}\Big|_{V_1} = e^2\frac{(\mu c)^{4w}}{(4\pi)^4}\frac{1}{w}\frac{8N_C}{9}\frac{m_\ell^2}{F^2} + \frac{\alpha}{4\pi}V_1\left[\frac{m_\pi^2}{3}\ell_\ell + m_\ell^2\left(\frac{5}{9}-\frac{4}{3}\ell_\ell\right)\right] \tag{55}$$

$$r_\ell^{e^2p^4}\Big|_{A_1} = -e^2\frac{(\mu c)^{4w}}{(4\pi)^4}\frac{1}{w}\frac{28}{3}\left(\bar{L}_9+\bar{L}_{10}\right)\frac{m_\ell^2}{F^2} + \frac{\alpha}{4\pi}A_1\left[-\frac{m_\pi^2}{3}\ell_\ell + m_\ell^2\left(\frac{13}{9}+\frac{7}{3}\ell_\ell\right)\right] \tag{56}$$

$$\begin{aligned} r_\ell^{e^2p^4}\Big|_{A_2} &= e^2\frac{(\mu c)^{4w}}{(4\pi)^4}\left[\frac{1}{w^2}+\frac{1}{w}\left(\frac{3}{2}+16\bar{L}_9\right)\right]\frac{m_\ell^2}{F^2} \\ &+ \frac{\alpha}{4\pi}\frac{m_\ell^2}{(4\pi F)^2}\left\{\left[\frac{13}{9}+8\,\bar{L}_9+\frac{16}{9}\left(\ell_\pi+\frac{1}{2}\ell_K\right)+\frac{2}{3}\left(\ell_\pi^2+\frac{1}{2}\ell_K^2\right)\right]\right. \\ &+ \left. 4\left(4\bar{L}_9-\frac{1}{6}-\frac{1}{3}\ell_\pi-\frac{1}{6}\ell_K\right)\ell_\ell - \frac{8}{9}\log(z_\ell\sqrt{\tilde{z}_\ell}) + f_1(z_\ell) + \frac{1}{2}f_1(\tilde{z}_\ell)\right\} \end{aligned} \tag{57}$$

$$
\left. r_\ell^{e^2p^4} \right|_{F_V} = -e^2 \frac{(\mu c)^{4w}}{(4\pi)^4} \left[\frac{1}{4\,w^2} + \frac{1}{w} \left(\frac{5}{12} + 4\bar{L}_9 \right) \right] \frac{m_\ell^2}{F^2}
$$

$$
+ \frac{\alpha}{4\pi} \frac{m_\ell^2}{(4\pi F)^2} \left\{ \left[-\frac{19}{36} - \left(\frac{1}{2} + 4\bar{L}_9 \right) \ell_\pi + \frac{1}{6}\ell_\pi^2 + \frac{1}{6}\ell_\pi \ell_K - \frac{1}{3}\ell_K - \frac{1}{12}\ell_K^2 \right] \right.
$$

$$
\left. + \frac{z_\ell}{1 - z_\ell} \log z_\ell \left(4\bar{L}_9 - \frac{1}{6} - \frac{1}{3}\ell_\pi - \frac{1}{6}\ell_K \right) + f_2(z_\ell) + f_3(\tilde{z}_\ell, \tilde{z}_\pi) \right\} . \tag{58}
$$

In terms of the building block functions $\tilde{E}_n(x)$, $\tilde{R}_n(x)$, $T^{\pi\pi}(x)$, and $T^{KK}(x, y)$ defined in Appendix A (Eqs. 129 and 149-151), the finite functions $f_{1,2,3}$ read:

$$
f_1(x) = \frac{97}{54} + \frac{4}{3} \left(4 \left(\tilde{R}_0(x) + \frac{1}{6} \log x \right) - \tilde{R}_1(x) - 4\tilde{R}_2(x) + \tilde{R}_3(x) \right)
$$

$$
+ \frac{1}{3} \left(-9\tilde{E}_0(x) + 6\tilde{E}_1(x) + 8\tilde{E}_2(x) - 6\tilde{E}_3(x) + \tilde{E}_4(x) \right) \tag{59}
$$

$$
f_2(x) = \frac{490 + 3(147 - 32\pi^2)}{108} + \frac{1}{3}T^{\pi\pi}(x) + \frac{1}{3}\frac{T^{\pi\pi}(x) - T^{\pi\pi}(0)}{x} + \tilde{R}_2(x) - \tilde{R}_0(x)
$$

$$
+ \frac{1}{1 - x} \left[x \left(\tilde{R}_2(x) - \tilde{R}_0(x) \right) + \frac{1}{3} x \left(\tilde{E}_2(x) - \tilde{E}_0(x) \right) \right.
$$

$$
\left. + x \left(\frac{540 + 3(147 - 32\pi^2)}{108} \right) + \frac{1}{3} \left(2(\tilde{E}_0(x) - \tilde{E}_2(x)) + \tilde{E}_3(x) - \tilde{E}_1(x) \right) \right] \tag{60}
$$

$$
f_3(x, y) = -\frac{25}{108} + \frac{1}{6}T^{KK}(x, y) + \frac{1}{6}\frac{T^{KK}(x, y) - T^{KK}(0, y)}{x/y}
$$

$$
+ \frac{1}{6}(4 - y) \left(\tilde{R}_2(x) - \tilde{R}_0(x) \right) + \frac{1}{3}\frac{y - 2}{y - x} \left(\tilde{E}_2(y) - \tilde{E}_0(y) \right)
$$

$$
+ \frac{1}{6}\frac{1}{1 - x/y} \left[\tilde{E}_3(x) - \tilde{E}_1(x) - \tilde{E}_3(y) + \tilde{E}_1(y) \right.
$$

$$
\left. + x/y\,(4 - y) \left(\tilde{R}_2(x) - \tilde{R}_0(x) \right) + (x/y - 2) \left(\tilde{E}_2(x) - \tilde{E}_0(x) \right) \right] \tag{61}
$$

Note that the functions $f_{1,2}(x)$ and $f_3(x, y)$ are non-singular for $x \to 0$ (corresponding to $m_\ell \to 0$).

In order to make the $O(e^2p^4)$ amplitude UV finite we introduce in the EFT a local counterterm. By power counting such a term cannot distinguish K and π decays. Its contribution to the amplitude is:

$$
\left. r_\ell^{e^2p^4} \right|_{CT} = e^2 \frac{m_\ell^2}{F^2} \frac{(\mu c)^{4w}}{(4\pi)^4} \left[\frac{d_2}{w^2} + \frac{d_1^{(0)} + d_1^{(L)}(\mu)}{w} + r_{CT}(\mu) \right] \tag{62}
$$

with

$$d_2 = -\frac{3}{4} \tag{63}$$

$$d_1^{(0)} = -\frac{15}{4} \tag{64}$$

$$d_1^{(L)}(\mu) = -\frac{8}{3}\bar{L}_9(\mu) + \frac{28}{3}\bar{L}_{10}(\mu) \ . \tag{65}$$

The finite coupling $r_{CT}(\mu)$ satisfies the following renormalization group equation:

$$\mu \frac{d}{d\mu} r_{CT}(\mu) = -\left(4\, d_1^{(0)} + 2\, d_1^{(L)}(\mu)\right) \ . \tag{66}$$

6 Matching

6.1 Strategy

Within ChPT, the loop calculation of $T_\ell^{e^2p^4}$ produces an ultraviolet divergence proportional to $(\alpha/\pi)m_\ell^2/(4\pi F)^2$, indicating the need to introduce a local operator of $O(e^2p^4)$, with an associated low-energy coupling. While the divergent part of the effective coupling is fully determined by our loop calculation, in order to estimate its finite part one needs to go beyond the low-energy effective theory and use information on the underlying QCD dynamics.

In full generality, the $O(\alpha)$ virtual-photon correction to the $\pi \to \ell\nu$ amplitude is given by a sum of contributions that share the following convolution structure:

$$T_\ell^{e^2p^4}\Big|_{QCD} = \int \frac{d^dq}{(2\pi)^d}\, K(q,p,p_e)\, \Pi_{QCD}(q^2,W^2)\,\Bigg|_{e^2p^4} , \tag{67}$$

where K is a known kernel, Π_{QCD} stands for one of the invariant form factors appearing in Eq. 24, and one has to expand the r.h.s. up to $O(e^2p^4)$ in the chiral power counting. In the framework of the low-energy effective theory, when calculating $T_\ell^{e^2p^4}$ we use the $O(p^4)$ ChPT representation for the form factors, $\Pi_{QCD} \to \Pi_{ChPT}^{p^4}$ in Eq. 67. While this representation is valid at scales below m_ρ (and generates the correct single- and double-logs upon integration) it leads to the incorrect UV behavior of the integrand in 67, which is dictated by the Operator Product Expansion (OPE) for the $\langle VVP\rangle$ and $\langle VAP\rangle$ correlators. As anticipated, this forces the introduction of a local operator of $O(e^2p^4)$ whose finite coupling is a priori unknown, so that:

$$T_\ell^{e^2p^4}\Big|_{ChPT} = \int \frac{d^dq}{(2\pi)^d}\, K(q,p,p_e)\, \Pi_{ChPT}^{p^4}(q^2,W^2) + T_\ell^{e^2p^4,CT} \ . \tag{68}$$

The physical matching condition $T_\ell^{e^2p^4}|_{ChPT} = T_\ell^{e^2p^4}|_{QCD}$ in principle allows one to determine the finite part of the counterterm. From the above discussion it is evident that the counterterm arises from the UV region in the convolution of Eq. 67, so in order to estimate it we need a suitable representation of the correlators which is valid for momenta beyond the chiral regime. This poses a complex non-perturbative problem that we are not able to solve within full QCD.

The problem becomes tractable if we work within the context of a truncated version of large-N_C QCD, in which we replace $\Pi_{QCD} \to \Pi_{QCD_\infty}$ and $\Pi_{ChPT} \to \Pi_{ChPT_\infty}$. In this framework we approximate the full QCD correlators by meromorphic functions, i.e. we assume that the correlators are saturated by the exchange of a *finite* number of narrow resonances (at large N_C one would have an infinite number of resonances). The relevant resonance couplings are fixed by requiring that the correlators obey suitable sets of QCD short-distance constraints [23] (see discussion in the next section for an assessment of the model-dependence). Correspondingly, in the chiral effective theory the correlators are obtained by considering only tree-level diagrams involving Goldstone modes, with the couplings of higher order operators (in our case L_9 and L_{10}) consistently determined by integrating out the resonance fields. In this framework we are able to perform all integrations analytically and we determine the local coupling by the matching condition:

$$\begin{aligned} T_\ell^{e^2p^4,CT} &= \int \frac{d^dq}{(2\pi)^d} \, K(q,p,p_e) \, \Pi_{QCD_\infty}(q^2,W^2) \Bigg|_{e^2p^4} \\ &- \int \frac{d^dq}{(2\pi)^d} \, K(q,p,p_e) \, \Pi^{p^4}_{ChPT_\infty}(q^2,W^2) \, . \end{aligned} \tag{69}$$

Note that since we are using the large-N_C representations for the QCD and ChPT form factors (Π_{QCD_∞} and $\Pi^{p^4}_{ChPT_\infty}$), our matching procedure is going to miss corrections to $c_3^{CT}(\mu)$ sub-leading in the $1/N_C$ expansion, which are responsible for the "double-log" scale dependence of the counterterm.

6.2 Meromorphic approximation for the form factors

In order to implement the program described above, we need a suitable representation of the hadronic correlator of Eq. 24, to be used in the convolution integral of Eq. 22. In Refs. [24, 25, 26, 27, 28] one can find analysis of the $\langle VVP \rangle$ and $\langle VAP \rangle$ Green Functions, describing them in terms of simple meromorphic functions that respect the constraints imposed at low-momentum transfer by chiral symmetry and at high momentum-transfer by the OPE. For correlators that are order parameters of spontaneous chiral symmetry breaking, such as $\langle VVP \rangle$ and $\langle VAP \rangle$, this is a sensible approximation well supported by a number of studies [23].

Using the LSZ reduction formula, it is simple to extract the form factors from the correlators of Refs. [26, 27]. Denoting M_V and M_A the masses of vector and axial-vector meson resonances, and using $W = p - q$, we find:

$$
\begin{aligned}
V_1(q^2, W^2) &= \frac{1}{6}\frac{2(q^2 - q\cdot p) - \frac{N_C M_V^4}{4\pi^2 F^2}}{(q^2 - M_V^2)(W^2 - M_V^2)} && (70)\\
A_1(q^2, W^2) &= \frac{M_V^2 - M_A^2 - b_2 q^2 - b_3 W^2}{(q^2 - M_V^2)(W^2 - M_A^2)} && (71)\\
A_2(q^2, W^2) &= \frac{-2\,M_A^2 - d_2 W^2}{(q^2 - M_V^2)(W^2 - M_A^2)} && (72)\\
A_3(q^2, W^2) &= -\frac{2 + d_2}{(q^2 - M_V^2)(W^2 - M_A^2)} && (73)\\
F_V(q^2) &= \frac{M_V^2}{M_V^2 - q^2} \ . && (74)
\end{aligned}
$$

To leading order in powers of q^2 and W^2, the above results reproduce the ChPT results to $O(p^4)$, Eqs. 27, 28, 29, 30, provided one identifies the low-energy constants with their resonance-saturated values $L_9 \to F^2/(2M_V^2)$ and $L_{10} \to -F^2/4(1/M_V^2 + 1/M_A^2)$, and provided one neglects the chiral loops. Note that $A_3(q^2, W^2)$ is not relevant for our matching procedure, since it starts to contribute to our amplitude to $O(e^2 p^6)$. The dimensionless constants $b_{2,3}$ and d_2 [27] are *a priori* unknnown and can be fixed by imposing constraints on the asymptotic behavior of the Green Functions. Different results exist in the literature, corresponding to different choices of the resonance content of the meromorphic ansatz and consequently different sets of QCD short-distance constraints. These different choices will allow us to quantify at least in part the model-dependence of the final answer. Let us briefly discuss the two choices:

1. The authors of Refs. [25, 26] include in their hadronic ansatz for the $\langle VAP \rangle$ correlator only the lowest lying V and A resonances and after imposing short-distance constraints they find
$$b_2 = \frac{1}{2} \ , \quad b_3 = -\frac{1}{2} \ , \quad d_2 = -1 \ . \qquad (75)$$

2. On the other hand, the authors of Ref. [27] include also one multiplet of pseudoscalar (P) resonances in the truncated spectrum. After imposing a larger set of short distance constraints they find $b_2 = 1, b_3 = 0, d_2 = 0$.
$$b_2 = 1 \ , \quad b_3 = 0 \ , \quad d_2 = 0 \ . \qquad (76)$$

For the present application it is crucial to check that the vertex functions $(\Gamma_{VV})^{abc}_{\mu\nu}(q,p) = \int d^4x \langle 0|T(V^a_\mu(x)\, V^b_\nu(0)|\pi^c(p)\rangle$ and $(\Gamma_{VA})^{abc}_{\mu\nu}(q,p) = \int d^4x \langle 0|T(V^a_\mu(x)\, A^b_\nu(0)|\pi^c(p)\rangle$ satisfy the correct asymptotic behavior dictated by QCD for $q \to \infty$. With our ansatz Γ_{VV} satisfies the leading and next-to-leading power behavior ($O(q^{-1})$ and $O(q^{-2})$) required by QCD. Concerning Γ_{AV}, the ansatz of Ref. [25, 26] reproduces the QCD behavior to $O(q^{-1})$ and $O(q^{-2})$. On the other hand, the ansatz of Ref. [27] has the correct $O(q^{-1})$ behavior but to $O(q^{-2})$ gives a result that is twice the QCD one. Due to these considerations, in our analysis we will use choice 1. above for the $\langle VAP \rangle$ form factors and use difference in the results from choice 2. as an indicator of the model dependence. We will find that the spread in result is minimal, showing that the convolution integral is dominated by low and intermediate virtualities. This feature is quite welcome in that it makes our results more robust.

6.3 Results

The matching calculation is straightforward but tedious. It involves (i) inserting the large-N_C form factors of in the convolution representation of Eq. 22; (ii) reducing the resulting integrals to scalar Passarino-Veltman functions; (iii) expanding the full result in powers of $m_{\ell,\pi}/M_V$, up to order $(m/M_V)^2$; (iv) finally, subtracting the ChPT$_\infty$ result from the expanded full result, thus obtaining the counterterm amplitude according to Eq. 69. The details of this calculation are reported in Appendix B.

Using the coefficients $b_{2,3}$ and d_2 as determined in Ref. [26], and defining $z_A = M_A/M_V$, we find

$$\begin{aligned} T_\ell^{e^2p^4,CT}(\mu) &= T_\ell^{p^2} \, \frac{\alpha}{4\pi} \frac{m_\ell^2}{M_V^2} \Bigg\{ \left[\frac{4}{3} V_1 \, M_V^2 - \frac{7}{3z_A^2} - \frac{11}{3} \right] \log \frac{M_V^2}{\mu^2} - \frac{19}{9} V_1 \, M_V^2 \\ &+ \frac{1}{18 z_A^2 \, (-1 + z_A^2)^2} \left[-37 + 31 z_A^2 - 17 z_A^4 + 11 z_A^6 \right] \\ &- \frac{2}{3 z_A^2 \, (-1 + z_A^2)^3} \left[-7 + 5 z_A^2 + z_A^4 - z_A^6 \right] \log z_A \Bigg\} \, . \end{aligned} \tag{77}$$

If one uses instead the values of $b_{2,3}$ and d_2 from Ref. [27], the counterterm amplitude is obtained by adding to Eq. 77 the following expression:

$$\delta T_\ell^{e^2p^4,CT} = T_\ell^{p^2} \, \frac{\alpha}{4\pi} \frac{m_\ell^2}{M_V^2} \, \frac{\log z_A^2}{3(-1 + z_A^2)} \, . \tag{78}$$

We defer a full discussion of the implications of this result to Section 8. Here we wish to point out that our matching procedure captures in full the "single-log" scale dependence of the counterterm as dictated by the renormalization group. This means that the scale dependence of Eq. 77 cancels the bulk of the scale dependence from chiral loops, leading to a very stable result.

7 Real-photon corrections

7.1 Radiative decay in ChPT

The amplitude for the radiative decay $\pi^+(p) \to \ell^+(p_\ell)\nu(p_\nu)\gamma(q)$ can be written as [20]:

$$
\begin{aligned}
T_\ell^{\mathrm{rad}} &= i\,2eG_F F_\pi V_{ud}^*\,\epsilon_\mu^*(q)\Big(B^\mu \;-\; H^{\mu\nu}\,l_\nu\Big) && (79)\\
B^\mu &= m_\ell\,\bar{u}_L(p_\nu)\left[\frac{2p^\mu}{2p\cdot q} - \frac{2p_\ell^\mu + \not{q}\gamma^\mu}{2p_\ell\cdot q}\right] v(p_\ell) && (80)\\
H^{\mu\nu} &= iV_1\epsilon^{\mu\nu\alpha\beta}q_\alpha p_\beta - A_1\Big(q\cdot(p-q)\,g^{\mu\nu} - (p-q)^\mu\,q^\nu\Big) && (81)\\
l_\nu &= \bar{u}_L(p_\nu)\gamma_\nu v(p_\ell)\ , && (82)
\end{aligned}
$$

with the form factors V_1 and A_1 given to $O(p^4)$ in Eqs. 27 and 28. The part of the amplitude proportional to B^μ is referred to as "Inner Bremsstrahlung" (IB) component, while the part proportional to $H^{\mu\nu}$ is called "Structure Dependent" (SD) component. IB and SD components are separately gauge invariant. The radiative decay rate has a term coming from the IB amplitude squared, a term from the interference of IB and SD, and finally a term proportional to the SD amplitude squared. To the order we work in the chiral expansion, only the first two terms have to be considered in principle, and lead, respectively, to $\delta_\ell^{e^2p^2}$ and $\delta_\ell^{e^2p^4}$ in the expression for $R_{e/\mu}$ in Eq. 9.

Introducing the dimensionless kinematical variables

$$
x = \frac{2p\cdot q}{m_\pi^2} \qquad y = \frac{2p\cdot p_\ell}{m_\pi^2}\ , \tag{83}
$$

the differential radiative decay rate is [20]

$$
\begin{aligned}
\frac{d^2\Gamma(\pi\to\ell\nu\gamma)}{dx\,dy} &= \frac{\alpha}{2\pi}\frac{\Gamma^{(0)}(\pi\to\ell\nu)}{(1-z_\ell)^2}\left[f_{IB}(x,y) + m_\pi^2\Big(V_1\,f_{INT}^{(V)}(x,y) + A_1\,f_{INT}^{(A)}(x,y)\Big)\right] && (84)\\
f_{IB}(x,y) &= \frac{1-y+z_\ell}{x^2(x+y-1-z_\ell)}\left[x^2 + 2(1-x)(1-z_\ell) - \frac{2xz_\ell(1-z_\ell)}{x+y-1-z_\ell}\right] && (85)\\
f_{INT}^{(V)}(x,y) &= \frac{x(1-y+z_\ell)}{x+y-1-z_\ell} && (86)\\
f_{INT}^{(A)}(x,y) &= \frac{1}{x}\,\frac{(1-y+z_\ell)}{x+y-1-z_\ell}\left[2z_\ell - x^2 + 2(1-x)(1-x-y)\right]\ . && (87)
\end{aligned}
$$

The total rates are obtained by integrating over the physical region

$$
\begin{aligned}
2\sqrt{z_\gamma} \leq\; & x \;\leq 1 - z_\ell + z_\gamma\\
1 - x + \frac{z_\ell}{1-x} \leq\; & y \;\leq 1 + z_\ell\ . && (88)
\end{aligned}
$$

When integrating the IB component over the whole physical region, an infrared divergence arises. It must be regulated in the same way as in the virtual corrections (in our choice by giving an infinitesimal mass to the photon), and it will eventually disappear when one calculates the observable inclusive rate. All integrals can be done analytically, and the results are reported in the next section.

7.2 Results for $\delta_\ell^{e^2p^2}$, $\delta_\ell^{e^2p^4}$, and $\delta_\ell^{e^2p^6}$

The IB contribution to the radiative rate reads [19] (recall the definition of $\delta_\ell^{e^2p^{2n}}$ in Eq. 7):

$$\begin{aligned}\delta_\ell^{e^2p^2} &= \frac{\alpha}{\pi}\Bigg\{ -\frac{z_\ell(10-7z_\ell)}{4(1-z_\ell)^2}\log z_\ell + \frac{15-21z_\ell}{8(1-z_\ell)} - 2\frac{1+z_\ell}{1-z_\ell}\,Li_2(1-z_\ell) \\ &+ \left[2+\frac{1+z_\ell}{1-z_\ell}\log z_\ell\right]\left[\log\sqrt{z_\gamma} - \log(1-z_\ell) - \frac{1}{4}\log z_\ell + \frac{3}{4}\right]\Bigg\}\,, \end{aligned} \tag{89}$$

where

$$Li_2(x) = -\int_0^x \frac{dt}{t}\,\log(1-t)\,. \tag{90}$$

The above formula refers to the fully photon-inclusive radiative rate. If one considers only the radiation of soft photons with $E_\gamma^{\rm CMS} < \omega \ll m_\pi$ one finds, up to terms suppressed by ω/m_π [29],

$$\begin{aligned}\delta_\ell^{e^2p^2}(\omega) &= \frac{\alpha}{\pi}\Bigg\{1 - \frac{1+z_\ell}{2(1-z_\ell)}\log z_\ell - \frac{1+z_\ell}{4(1-z_\ell)}\log^2 z_\ell - \frac{1+z_\ell}{1-z_\ell}\,Li_2(1-z_\ell) \\ &+ \left[2+\frac{1+z_\ell}{1-z_\ell}\log z_\ell\right]\log\frac{m_\gamma}{2\omega}\Bigg\}\,, \end{aligned} \tag{91}$$

The interference between IB and SD amplitude (parameterized in terms of the form factors V_1 and A_1) reads:

$$\begin{aligned}\delta_\ell^{e^2p^4} &= \frac{\alpha}{2\pi}\frac{m_\pi^2}{(1-z_\ell)^2}\Bigg\{V_1\left[-\frac{17}{18}+\frac{z_\ell}{2}+\frac{z_\ell^2}{2}-\frac{z_\ell^3}{18}-\frac{1}{3}\log z_\ell - z_\ell\log z_\ell\right] \\ &+ A_1\left[\frac{7}{9}-2z_\ell+z_\ell^2+\frac{2\,z_\ell^3}{9}+\frac{1}{3}\log z_\ell - z_\ell^2\log z_\ell\right]\Bigg\}\,. \end{aligned} \tag{92}$$

Classifying the various terms according to their behavior with the lepton mass, one obtains:

$$\begin{aligned}\delta_\ell^{e^2p^4} &= \frac{\alpha}{2\pi}\left(\frac{7}{9}A_1 - \frac{17}{18}V_1\right)m_\pi^2 + \frac{\alpha}{2\pi}\left(A_1 - V_1\right)\frac{m_\pi^2}{3}\log z_\ell \\ &+ \frac{\alpha}{2\pi}\left(-\frac{4}{9}A_1 - \frac{25}{18}V_1\right)m_\ell^2 + \frac{\alpha}{2\pi}\left(2A_1 - 5V_1\right)\frac{m_\ell^2}{3}\log z_\ell \\ &+ \frac{\alpha}{2\pi}m_\ell^2\frac{z_\ell}{(1-z_\ell)^2}\Bigg\{-\frac{2}{3}A_1\Big(1-z_\ell+z_\ell\log z_\ell\Big) \\ &- \frac{1}{3}V_1\Big[4(1-z_\ell)+(9-5z_\ell)\log z_\ell\Big]\Bigg\}\,. \end{aligned} \tag{93}$$

Finally, we report here the purely SD contribution to the radiative rate, which is down by one order in the chiral expansion but does not suffer from helicity suppression. We find:

$$\begin{aligned}\delta_\ell^{e^2p^6} &= \frac{\alpha}{8\pi}\, m_\pi^4 \left(V_1^2 + A_1^2\right) \left[\frac{1}{30\, z_\ell} - \frac{11}{60}\right. \\ &+ \left.\frac{z_\ell}{20(1-z_\ell)^2}\left(12 - 3z_\ell - 10 z_\ell^2 + z_\ell^3 + 20\, z_\ell \log z_\ell\right)\right] . \end{aligned} \quad (94)$$

8 Phenomenology of $R_{e/\mu}$

We now put together all the results obtained so far. The starting point of our phenomenological analysis of $R_{e/\mu}^{(\pi,K)}$ is Eq. 8, which organizes the electroweak corrections to the leading order result of Eq. 3 according to the chiral expansion. Incorporating the effects of leading higher order logs [7] of the form $\alpha^n \log^n(m_\mu/m_e)$ through the correction Δ_{LL}, we can write Eq. 8 as:

$$R_{e/\mu}^{(P)} = R_{e/\mu}^{(0),(P)} \left[1 + \Delta_{e^2p^2}^{(P)} + \Delta_{e^2p^4}^{(P)} + \Delta_{e^2p^6}^{(P)} + ...\right]\left[1 + \Delta_{LL}\right] \quad (95)$$

The leading electromagnetic correction in ChPT corresponds to the point-like approximation for pion and kaon [7, 11, 19]:

$$\Delta_{e^2p^2}^{(P)} = \frac{\alpha}{\pi}\left[F(\frac{m_e^2}{m_P^2}) - F(\frac{m_\mu^2}{m_P^2})\right] \quad (96)$$

$$\begin{aligned} F(z) &= \frac{3}{2}\log z + \frac{13-19z}{8(1-z)} - \frac{8-5z}{4(1-z)^2}\, z\log z - \left(2 + \frac{1+z}{1-z}\log z\right)\log(1-z) \\ &- 2\frac{1+z}{1-z}\, Li_2(1-z)\ . \end{aligned} \quad (97)$$

The structure dependent effects are all contained in $\Delta_{e^2p^4}$ and higher order terms, which are the main subject of this work. Neglecting terms of order $(m_e/m_\rho)^2$, the most general parameterization of the NLO ChPT contribution can be written in the form

$$\Delta_{e^2p^4}^{(P)} = \frac{\alpha}{\pi}\frac{m_\mu^2}{m_\rho^2}\left(c_2^{(P)}\log\frac{m_\rho^2}{m_\mu^2} + c_3^{(P)} + c_4^{(P)}(m_\mu/m_P)\right) + \frac{\alpha}{\pi}\frac{m_P^2}{m_\rho^2}\,\tilde{c}_2^{(P)}\log\frac{m_\mu^2}{m_e^2}\ , \quad (98)$$

which highlights the dependence on lepton masses. The dimensionless constants $c_{2,3}^{(P)}$ do not depend on the lepton mass but depend logarithmically on hadronic masses, while $c_4^{(P)}(m_\mu/m_P) \to 0$ as $m_\mu \to 0$. (Note that our $c_{2,3}^{(\pi)}$ do not coincide with $C_{2,3}$ of Ref. [7], because their C_3 is not constrained to be m_ℓ-independent.)

Finally, let us note that the results for $c_{2,3,4}^{(P)}$ and $\tilde{c}_2^{(P)}$ depend on the definition of the inclusive rate $\Gamma(P \to \ell\bar{\nu}_\ell[\gamma])$. The radiative amplitude is the sum of the inner bremsstrahlung (T_{IB}) component of $O(ep)$ and a structure dependent (T_{SD}) component of $O(ep^3)$ [20]. The experimental definition of $R_{e/\mu}^{(\pi)}$ is fully inclusive on the radiative mode, so that $\Delta_{e^2p^4}^{(\pi)}$ receives a contribution from the interference of T_{IB} and T_{SD}. Moreover, in this case one also has to include the effect of $\Delta_{e^2p^6}^{(\pi)} \propto |T_{SD}|^2$, that is formally of $O(e^2p^6)$, but is not helicity suppressed and behaves as $\Delta_{e^2p^6} \sim \alpha/\pi\,(m_P/M_V)^4\,(m_P/m_e)^2$. On the other hand, the usual experimental definition of $R_{e/\mu}^{(K)}$ is not fully inclusive on the radiative mode. It corresponds to including the effect of T_{IB} in $\Delta_{e^2p^2}^{(K)}$ (dominated by soft photons) and excluding altogether the effect of T_{SD}: consequently $c_n^{(\pi)} \neq c_n^{(K)}$.

8.1 Results for $R_{e/\mu}^{(\pi)}$

Recalling the definitions $\bar{L}_9 \equiv (4\pi)^2 L_9^r(\mu)$, $\ell_P \equiv \log(m_P^2/\mu^2)$ (μ is the chiral renormalization scale), $\gamma \equiv A_1(0,0)/V_1(0,0)$, $z_\ell \equiv (m_\ell/m_\pi)^2$, we find:

$$c_2^{(\pi)} = \frac{2}{3}\, m_\rho^2\, \langle r^2\rangle_V^{(\pi)} + 3\,(1-\gamma)\,\frac{m_\rho^2}{(4\pi F)^2} \qquad \tilde{c}_2^{(\pi)} \;=\; 0 \tag{99}$$

$$\begin{aligned} c_3^{(\pi)} = -\frac{m_\rho^2}{(4\pi F)^2}\Bigg[& \frac{31}{24} - \gamma + 4\,\bar{L}_9 + \left(\frac{23}{36} - 2\,\bar{L}_9 + \frac{1}{12}\ell_K\right)\ell_\pi + \frac{5}{12}\ell_\pi^2 + \frac{5}{18}\ell_K + \frac{1}{8}\ell_K^2 \\ & + \left(\frac{5}{3} - \frac{2}{3}\gamma\right)\log\frac{m_\rho^2}{m_\pi^2} + \left(2 + 2\,\kappa^{(\pi)} - \frac{7}{3}\gamma\right)\log\frac{m_\rho^2}{\mu^2} + K^{(\pi)}(0)\Bigg] \;+\; c_3^{CT}(\mu) \end{aligned} \tag{100}$$

$$\begin{aligned} c_4^{(\pi)}(m_\ell) = -\frac{m_\rho^2}{(4\pi F)^2}\Bigg\{ & \frac{z_\ell}{3(1-z_\ell)^2}\left[\Big(4(1-z_\ell) + (9-5z_\ell)\log z_\ell\Big) + 2\,\gamma\,\Big(1 - z_\ell + z_\ell \log z_\ell\Big)\right] \\ & + \left(\kappa^{(\pi)} + \frac{1}{3}\right)\frac{z_\ell}{2(1-z_\ell)}\,\log z_\ell + K^{(\pi)}(m_\ell) - K^{(\pi)}(0)\Bigg\} \end{aligned} \tag{101}$$

where $\kappa^{(\pi)}$ is related to the $O(p^4)$ pion charge radius by:

$$\kappa^{(\pi)} \equiv 4\,\bar{L}_9 - \frac{1}{6}\ell_K - \frac{1}{3}\ell_\pi - \frac{1}{2} = \frac{(4\pi F_\pi)^2}{3}\,\langle r^2\rangle_V^{(\pi)}\ . \tag{102}$$

In the above equations we have used the definition:

$$K^{(\pi)}(m_\ell) = \frac{1}{2}\left[f_1(z_\ell) + \frac{1}{2}f_1(\tilde{z}_\ell) + f_2(z_\ell) + f_3(\tilde{z}_\ell, \tilde{z}_\pi) - \frac{8}{9}\log\frac{m_\rho^2}{m_\pi^2} - \frac{4}{9}\log\frac{m_\rho^2}{m_K^2}\right]\ . \tag{103}$$

The function $K^{(\pi)}(m_\ell)$ does not contain any large logarithms ($K^{(\pi)}(m_\mu) = -0.025$ and $K^{(\pi)}(0) = 0.085$) and gives a small fractional contribution to $c_{3,4}^{(\pi)}$.

Full numerical values of $c_{2,3,4}^{(\pi)}$ and $\tilde{c}_2^{(\pi)}$ are reported in Table 1, with uncertainties due to matching procedure and input parameters (L_9 and γ [30]). We now discuss the results obtained and make contact with the previous literature.

- We find $\tilde{c}_2^{(\pi)} = 0$ in accordance to a theorem by Marciano and Sirlin [31]. This result arises from an exact cancellation of virtual photon contributions proportional to V_1

	$(P=\pi)$	$(P=K)$
$\tilde{c}_2^{(P)}$	0	$(7.84 \pm 0.07_{\gamma}) \times 10^{-2}$
$c_2^{(P)}$	$5.2 \pm 0.4_{L_9} \pm 0.01_{\gamma}$	$4.3 \pm 0.4_{L_9} \pm 0.01_{\gamma}$
$c_3^{(P)}$	$-10.5 \pm 2.3_{\rm m} \pm 0.53_{L_9}$	$-4.73 \pm 2.3_{\rm m} \pm 0.28_{L_9}$
$c_4^{(P)}(m_\mu)$	$1.69 \pm 0.07_{L_9}$	$0.22 \pm 0.01_{L_9}$

Table 1: Numerical values for $c_n^{(P)}$ of Eq. 98, for $P = \pi, K$. The uncertainties correspond to the input values $L_9^r(\mu = m_\rho) = (6.9 \pm 0.7) \times 10^{-3}$, $\gamma = 0.465 \pm 0.005$ [30], and to the matching procedure (m), affecting only $c_3^{(P)}$.

and A_1 and real photon contribution due to the interference of SD and IB amplitudes. The cancellation occurs only when the fully inclusive rate is considered.

- The coefficient $c_2^{(\pi)}$ is a parameter-free prediction of ChPT to this order. It involves the $O(p^4)$ LECs L_9 and L_{10}, related to the pion charge radius and the ratio of axial-to-vector form factors γ measurable in the radiative pion and kaon decay.

- The coefficient $c_3^{(\pi)}$ receives a predictable contribution from loops in the ChPT framework, as well as a local contribution that cannot be predicted in the purely EFT approach, denoted by $c_3^{CT}(\mu)$. Both contributions are renormalization-scale dependent, while the sum is not. $c_3^{CT}(\mu)$ is related to the low energy coupling $r_{CT}(\mu)$ introduced in Eq. 62 by $r_{CT}(\mu) = -2(4\pi F)^2/m_\rho^2 \, c_3^{CT}(\mu)$. Our matching procedure gives for the counterterm ($z_A \equiv (M_A/M_V)$ and taking $M_V = m_\rho$):

$$\begin{aligned} c_3^{CT}(\mu) &= -\frac{19\, m_\rho^2}{9(4\pi F)^2} + \left(\frac{4\, m_\rho^2}{3(4\pi F)^2} + \frac{7 + 11 z_A^2}{6 z_A^2}\right) \log \frac{m_\rho^2}{\mu^2} \\ &+ \frac{37 - 31 z_A^2 + 17 z_A^4 - 11 z_A^6}{36 z_A^2 (1 - z_A^2)^2} \\ &- \frac{7 - 5 z_A^2 - z_A^4 + z_A^6}{3 z_A^2 (-1 + z_A^2)^3} \log z_A \ . \end{aligned} \tag{104}$$

Numerically, using $z_A = \sqrt{2}$ [32], we find $c_3^{CT}(m_\rho) = -1.61$, implying that the counterterm induces a sub-leading correction to c_3 (see Table 1). The model dependence due to different choices of the hadronic ansatz (Ref. [26] vs Ref. [27]) is negligible, being $\Delta c_3^{CT} = 0.12$. The scale dependence of $c_3^{CT}(\mu)$ partially cancels the scale dependence of the chiral loops (our procedure captures all the "single-log" scale dependence). Taking a very conservative attitude we assign to c_3 an uncertainty equal to 100% of the local contribution ($|\Delta c_3| \sim 1.6$) plus the effect of residual renormalization scale dependence, obtained by varying the scale μ in the range $0.5 \to 1$ GeV ($|\Delta c_3| \sim 0.7$), leading to $\Delta c_3^{(\pi,K)} = \pm 2.3$.

- Finally, the coefficient $c_4^{(\pi)}$ can be calculated in terms of the LEC L_9 and the lepton and meson masses and decay constants. Not surprisingly we find that this effect is only marginally important.

As a check on our calculation, we have verified that if we neglect c_3^{CT} and pure two-loop effects, and if we use $L_9 = F^2/(2M_V^2)$ (vector meson dominance), our results for $c_{2,3,4}^{(\pi)}$ are fully consistent with previous analyses of the leading structure dependent corrections based on current algebra [7, 9]. Moreover, our numerical value of $\Delta_{e^2p^4}^{(\pi)}$ reported in Table 2 is very close to the corresponding result in Ref. [7], namely $\Delta_{e^2p^4}^{(\pi)} = (0.054 \pm 0.044) \times 10^{-2}$ [7] versus $\Delta_{e^2p^4}^{(\pi)} = (0.053 \pm 0.011) \times 10^{-2}$ (this work). Therefore, as far as $R_{e/\mu}^{(\pi)}$ is concerned, the net effect of our calculation is a reduction of the uncertainty by a factor of four.

8.2 Results for $R_{e/\mu}^{(K)}$

In the case of K decays we find:

$$c_2^{(K)} = \frac{2}{3}\, m_\rho^2 \, \langle r^2 \rangle_V^{(K)} + \frac{4}{3} \left(1 - \frac{7}{4} \gamma \right) \frac{m_\rho^2}{(4\pi F)^2} \tag{105}$$

$$\tilde{c}_2^{(K)} = \frac{1}{3}\,(1-\gamma)\, \frac{m_\rho^2}{(4\pi F)^2} \tag{106}$$

$$\begin{aligned} c_3^{(K)} = & -\frac{m_\rho^2}{(4\pi F)^2} \Bigg[-\frac{7}{72} - \frac{13}{9}\gamma + 4\,\bar{L}_9 + \left(\frac{23}{36} - 2\bar{L}_9 + \frac{1}{12}\ell_\pi \right) \ell_K + \frac{5}{12}\ell_K^2 + \frac{5}{18}\ell_\pi + \frac{1}{8}\ell_\pi^2 \\ & + \left(2 + 2\,\kappa^{(K)} - \frac{7}{3}\gamma \right) \log \frac{m_\rho^2}{\mu^2} + K^{(K)}(0) \Bigg] \; + \; c_3^{CT}(\mu) \end{aligned} \tag{107}$$

$$c_4^{(K)}(m_\ell) = -\frac{m_\rho^2}{(4\pi F)^2} \left\{ \left(\kappa^{(K)} + \frac{1}{3} \right) \frac{\tilde{z}_\ell}{2(1-\tilde{z}_\ell)} \, \log \tilde{z}_\ell + K^{(K)}(m_\ell) - K^{(K)}(0) \right\} \; , \tag{108}$$

where $\langle r^2 \rangle_V^{(K)}$ is the $O(p^4)$ kaon charge radius and

$$\kappa^{(K)} \equiv 4\,\bar{L}_9 - \frac{1}{6}\ell_\pi - \frac{1}{3}\ell_K - \frac{1}{2} = \frac{(4\pi F)^2}{3} \, \langle r^2 \rangle_V^{(K)} \; . \tag{109}$$

Moreover the function $K^{(K)}(m_\ell)$ is given by:

$$K^{(K)}(m_\ell) = \frac{1}{2} \left[f_1(\tilde{z}_\ell) + \frac{1}{2} f_1(z_\ell) + f_2(\tilde{z}_\ell) + f_3(z_\ell, 1/\tilde{z}_\pi) - \frac{8}{9} \log \frac{m_\rho^2}{m_K^2} - \frac{4}{9} \log \frac{m_\rho^2}{m_\pi^2} \right] \; . \tag{110}$$

As in the pion case, the function $K^{(K)}(m_\ell)$ does not contain any large logarithms ($K^{(K)}(m_\mu) = 0.93$ and $K^{(\pi)}(0) = 1.05$) and gives a small fractional contribution to $c_{3,4}^K$.

Note that apart from missing contributions from the SD radiation, the $c_{2,3,4}^{(K)}$ and $\tilde{c}_2^{(K)}$ are obtained from the $c_{2,3,4}^{(\pi)}$ and $\tilde{c}_2^{(\pi)}$ by interchanging m_π with m_K everywhere (the underlying reason is given in Sect. 4.4). The numerical values of $c_{2,3,4}^{(K)}$ and $\tilde{c}_2^{(K)}$ are reported in Table 1.

	$(P=\pi)$	$(P=K)$
$\Delta^{(P)}_{e^2p^2}$ (%)	-3.929	-3.786
$\Delta^{(P)}_{e^2p^4}$ (%)	0.053 ± 0.011	0.135 ± 0.011
$\Delta^{(P)}_{e^2p^6}$ (%)	0.073	
Δ_{LL} (%)	0.055	0.055

Table 2: Numerical summary of various electroweak corrections to $R^{(\pi,K)}_{e/\mu}$. The uncertainty in $\Delta_{e^2p^4}$ corresponds to the matching procedure.

8.3 Resumming long distance logarithms

At the level of uncertainty considered, one needs to include higher order long distance corrections [7], generalizing the leading contribution $\Delta_{e^2p^2} \sim -3\alpha/\pi \log m_\mu/m_e \sim -3.7\%$. The leading logarithms can be summed via the renormalization group and their effect amounts to multiplying $R^{(P)}_{e/\mu}$ by $1+\Delta_{LL}$, with [7]

$$1+\Delta_{LL} = \frac{\left(1-\frac{2}{3}\frac{\alpha}{\pi}\log\frac{m_\mu}{m_e}\right)^{9/2}}{1-\frac{3\alpha}{\pi}\log\frac{m_\mu}{m_e}} = 1.00055 \ . \tag{111}$$

8.4 Discussion

In Table 2 we summarize the various electroweak corrections to $R^{(\pi,K)}_{e/\mu}$. Applying these we arrive to our final results:

$$R^{(\pi)}_{e/\mu} = (1.2352 \pm 0.0001) \times 10^{-4} \tag{112}$$

$$R^{(K)}_{e/\mu} = (2.477 \pm 0.001) \times 10^{-5} \ . \tag{113}$$

The uncertainty we quote for $R^{(\pi)}_{e/\mu}$ is entirely induced by our matching procedure. However, in the case of $R^{(K)}_{e/\mu}$ we have inflated the nominal uncertainty arising from matching by a factor of four, to account for higher order chiral corrections, that are expected to scale as $\Delta_{e^2p^4} \times m_K^2/(4\pi F)^2$.

Our results have to be compared with the ones of Refs. [7] and [8], which we report in Table 3. While $R^{(\pi)}_{e/\mu}$ is in good agreement with both previous results, there is a discrepancy in $R^{(K)}_{e/\mu}$ that goes well outside the estimated theoretical uncertainties. We have traced back this difference to two problematic aspects of Ref. [8]. (i) The leading log correction Δ_{LL} is included with the wrong sign: this accounts for half of the discrepancy. (ii) The remaining effect is due to the difference in the NLO virtual correction, for which Finkemeier finds $\Delta^{(K)}_{e^2p^4} = 0.058\%$. We have serious doubts on the reliability of this number because the hadronic form factors modeled in Ref. [8] do not satisfy the correct QCD short-distance

	$10^4 \cdot R_{e/\mu}^{(\pi)}$	$10^5 \cdot R_{e/\mu}^{(K)}$
This work	1.2352 ± 0.0001	2.477 ± 0.001
Ref. [7]	1.2352 ± 0.0005	
Ref. [8]	1.2354 ± 0.0002	2.472 ± 0.001

Table 3: Comparison of our result with the most recent predictions of $R_{e/\mu}^{(\pi,K)}$.

behavior. At high momentum they fall off faster than the QCD requirement, thus leading to a smaller value of $\Delta_{e^2p^4}^{(K)}$ compared to our work.

9 The individual $\pi(K) \to \ell\bar{\nu}_\ell$ modes

The approach followed in this work is designed to obtain the ratio of $\pi(K) \to e\bar{\nu}_e$ and $\pi(K) \to \mu\bar{\nu}_\mu$ decay rates, because we have neglected all the Feynman diagrams in which the photon does not connect to the charged lepton. Including these diagrams in ChPT would generate new finite parts and UV divergences, and the corresponding local couplings would have to be evaluated within the $1/N_C$ expansion described earlier. We leave this task for possible future work.

However, despite the fact that we have not performed a full $O(e^2p^4)$ calculation of $\pi(K) \to \ell\bar{\nu}_\ell$, our results can still be used to update the theoretical analysis of these individual decay modes. Here we closely follow the analysis of Ref. [7]. Including all known short- and long-distance electroweak corrections, and parameterizing the hadronic effects in terms of a few dimensionless coefficients, the inclusive $P \to \ell\bar{\nu}_\ell[\gamma]$ decay rate $\Gamma_{P_{\ell 2[\gamma]}}$ can be written as:

$$\begin{aligned}\Gamma_{P_{\ell 2[\gamma]}} = \; & \Gamma^{(0)} \times \left\{1 + \frac{2\,\alpha}{\pi}\log\frac{m_Z}{m_\rho}\right\} \times \left\{1 + \frac{\alpha}{\pi}\,F(m_\ell^2/m_P^2)\right\} \times \left\{1 - \frac{\alpha}{\pi}\left[\frac{3}{2}\log\frac{m_\rho}{m_P}\right.\right. \\ + \; & \left.\left. c_1^{(P)} + \frac{m_\ell^2}{m_\rho^2}\left(c_2^{(P)}\log\frac{m_\rho^2}{m_\ell^2} + c_3^{(P)} + c_4^{(P)}(m_\ell/m_P)\right) - \frac{m_P^2}{m_\rho^2}\,\tilde{c}_2^{(P)}\log\frac{m_\rho^2}{m_\ell^2}\right]\right\}, \quad (114)\end{aligned}$$

where $\Gamma^{(0)}$ is the rate in absence of radiative corrections (see Eq. 5), the first bracketed term is the universal short distance electroweak correction, the second bracketed term is the universal long distance correction (point-like meson), and the third bracketed term parameterizes the effects of hadronic structure. The function $F(z)$ and the constants $c_{2,3,4}^{(P)}$ (and $\tilde{c}_2^{(P)}$) already appear in $R_{e/\mu}^{(P)}$ and their expressions and numerical values have been reported in the previous section. The only additional ingredient needed to predict the individual rates $\Gamma_{P_{\ell 2[\gamma]}}$ is the structure-dependent coefficient $c_1^{(P)}$, which does not depend on the lepton mass and starts at $O(e^2p^2)$ in ChPT. The explicit form (for both $P = \pi, K$) is given in [11] (Eqs. 5.11 and 5.14) and it depends on a combination of EM LECs of

$O(e^2p^2)$. These have been recently estimated in Ref. [33] in the same large-N_C framework adopted here, with the final result:

$$c_1^{(\pi)} = -2.56 \pm 0.5 \quad (115)$$
$$c_1^{(K)} = -1.98 \pm 0.5 \; . \quad (116)$$

So at the moment all the structure dependent coefficients $c_n^{(p)}$ are known to leading order in their expansion within the chiral effective theory (which is $O(e^2p^2)$ for $c_1^{(P)}$ and $O(e^2p^4)$ for the other coefficients) . The $O(e^2p^4)$ contribution to $c_1^{(P)}$ has not yet been calculated (this could be done by employing the techniques presented in this paper). On the basis of power counting we expect $c_1^{(P)}|_{e^2p^4} \lesssim 0.5$, which is consistent with the uncertainty assigned to $c_1^{(P)}$ [33].

Finally, we discuss here a quantity of interest in the experimental analysis of $K_{e2}/K_{\mu2}$, namely the $K \to \ell\nu$ rate with inclusion of only soft photons ($\omega \ll m_K$):

$$\Gamma_{K_{\ell2[\gamma]}}(\omega) \equiv \Gamma(K \to \ell\bar{\nu}_\ell) + \Gamma(K \to \ell\bar{\nu}_\ell\gamma)\Big|_{E_\gamma^{\rm CMS} < \omega} \; . \quad (117)$$

Using our results on the emission of soft photons (Eq. 91), it is simple to show that $\Gamma_{K_{\ell2[\gamma]}}(\omega)$ is given by Eq. 114 provided one replaces $F(z) \to F^{\rm soft}(z;\omega)$, with ($z = m_\ell^2/m_K^2$):

$$\begin{aligned} F^{\rm soft}(z;\omega) &= -\frac{3}{4} + \frac{3}{4}\log z - \frac{2\,z}{1-z}\log z - \frac{1+z}{1-z} Li_2(1-z) \\ &- \left[2 + \frac{1+z}{1-z}\log z\right] \log\frac{2\omega}{m_K} \; . \end{aligned} \quad (118)$$

10 Conclusions

In conclusion, by performing the first ChPT calculation to $O(e^2p^4)$ and a matching calculation of the relevant low energy coupling, we have improved the reliability of both the central value and the uncertainty of the ratios $R_{e/\mu}^{(\pi,K)}$. Our final result for $R_{e/\mu}^{(\pi)}$ is consistent with the previous literature, while we find a discrepancy in $R_{e/\mu}^{(K)}$, which we have traced back to inconsistencies in the analysis of Ref. [8]. Our results provide a clean basis to detect or constrain non-standard physics in these modes by comparison with upcoming experimental measurements.

As a byproduct of our main analysis, we also updated the expressions for the radiative corrections to the individual $\pi(K) \to \ell\bar{\nu}_\ell$ modes, which can be used to extract from experiment the combinations $F_\pi V_{ud}$ and $F_K V_{us}$.

Finally, it is worth mentioning that the ideas and techniques discussed in this article can be applied (i) to perform a full $O(e^2p^4)$ analysis of the individual $\pi(K) \to \ell\bar{\nu}_\ell$ modes; (ii) to deal with other processes that involve one pseudo-scalar meson and a lepton pair, such as $\tau \to K\nu_\tau[\gamma]$. In this case chiral effective theory techniques are not adequate, but the calculation based on the large-N_C representation for the $\langle VAP\rangle$ and $\langle VVP\rangle$ remains adequate.

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A Two-loop integrals

A.1 Procedure

In order to calculate the genuine two-loop integrals listed in Sect. 4.3 above, we use the d-dimensional dispersive representation of the function $\bar{J}^{aa}(q^2)$ [21, 22], which is easily derived from Eq. 36. Re-expressing all dimensionful parameters in units of m_a, one obtains:

$$\bar{J}^{aa}(q^2) = -m_a^{2w}\bar{q}^2 \int_4^\infty \frac{[ds]}{s}\frac{1}{(\bar{q}^2 - s)} \tag{119}$$

$$[ds] = \frac{ds}{(4\pi)^{2+w}}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+w\right)}\left(\frac{s}{4}-1\right)^w\left(1-\frac{4}{s}\right)^{1/2}, \tag{120}$$

where $\bar{q} = q/m_a$ and s are dimensionless variables. Upon inserting the representation of Eq. 119 in the expression for I_n^{aa} one immediately sees that the calculation is naturally separated in two steps: (i) a one-loop diagram involving one propagator of mass s; (ii) integration of the result over the variable s, with measure given by $[ds]/s$.

In order to exemplify the procedure, we report here the calculation of $I_1^{(\ell)\pi\pi}$. The other integrals can be worked out with similar techniques. We have found extremely useful the results of Ref. [22]. The case considered in that paper is slightly easier, because they only have one mass scale in the loops (m_π), while we have two.

Inserting the representation of Eq. 119 in the definition of $I_1^{(\ell)\pi\pi}$, and re-expressing *all* momentum variables in units of m_π, one arrives at (recall $d = 4 + 2w$):

$$I_1^{(\ell)\pi\pi} = -m_\pi^{4w}\int_4^\infty \frac{[ds]}{s}\int \frac{d^dq}{(2\pi)^d}\frac{1}{q^2 - s}\frac{1}{[(q-p_\ell)^2 - z_\ell]} \tag{121}$$

Here q and p_ℓ are dimensionless momentum variables (to avoid clutter we are not using the $\bar{q}, \bar{p_\ell}$ notation) and $z_\ell = (m_\ell/m_\pi)^2$. Combining the two denominators with the usual trick one gets:

$$
\begin{aligned}
I_1^{(\ell)\pi\pi} &= -m_\pi^{4w} \int_4^\infty \frac{[ds]}{s} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left[(q - x p_\ell)^2 - z(x,s)\right]^2} \\
&= -i\, m_\pi^{4w} \int_4^\infty \frac{[ds]}{s} \int_0^1 dx\, F_2[z(x,s)] \qquad (122)
\end{aligned}
$$

where $z(x,s) = z_\ell\, x^2 + s(1-x)$ and [22]

$$
i(-1)^n F_n[z] = \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left[q^2 - z\right]^n} \qquad (123)
$$

Explicitly one has

$$
F_n[z] = C(w)\, z^{w+2-n} \frac{\Gamma(n-2-w)}{\Gamma(n)}\,, \qquad n \geq 1 \qquad (124)
$$

with $C(w) = 1/(4\pi)^{2+w}$. Most of the non-trivial integrals that we need to calculate have the structure of Eq. 122. In order to make progress one needs to identify in Eq. 122 the finite part and divergent part. This is accomplished by using a set of recursion relations that are the subject of next subsection.

A.2 Recursion relations

In close analogy with Ref. [22] one can define (for m, n integers) :

$$
\tilde{E}(m,n;z_\ell) = \int_4^\infty \frac{[ds]}{s} \int_0^1 dx\, (1-x)^m\, F_n[z(x,s)] \qquad (125)
$$

where $z(x,s) = z_\ell\, x^2 + s(1-x)$. If $z_\ell \to 1$ then $\tilde{E}(m,n;z_\ell) \to E(m,n)$ defined in Appendix C of Ref. [22]. By use of integration by parts in the variable x and recalling the explicit form of $F_n[z]$ (Eq. 124), one can derive the following useful recursion relation:

$$
\begin{aligned}
(3+w+m-n)\, \tilde{E}(m,n;z_\ell) &= \frac{\Gamma(n-2-w)}{\Gamma(n)\Gamma(-w)}\, Q(w+1-n) \\
&- n\, z_\ell \left(\tilde{E}(m,n+1;z_\ell) - \tilde{E}(m+2,n+1;z_\ell) \right) \qquad (126)
\end{aligned}
$$

with

$$
\begin{aligned}
Q(\alpha) &= C(w)\, \Gamma(-w) \int_4^\infty [ds]\, s^\alpha \\
&= C^2(w)\, \Gamma(-w)\, \Gamma(-1-w-\alpha)\, \frac{\Gamma(-\alpha)}{\Gamma(-2\alpha)}\,. \qquad (127)
\end{aligned}
$$

Reassuringly, by setting $z_\ell = 1$ one recovers the result of Gasser-Sainio [22].

Eq. 126 is useful because it allows one to express the integrals $\tilde{E}(m, n \leq 2; z_\ell)$ in terms of known divergent quantities ($Q(\alpha)$) and the convergent integrals $\tilde{E}(m, 3 : z_\ell)$. Finally, let us provide an integral representation for $\tilde{E}(m, 3; z_\ell)$ (obtained by setting $d = 4$):

$$z_\ell \, \tilde{E}(m, 3; z_\ell) = \frac{1}{2(4\pi)^4} \, \tilde{E}_m(z_\ell) \tag{128}$$

$$\begin{aligned} \tilde{E}_m(z_\ell) &= z_\ell \int_4^\infty \frac{ds}{s} \left(1 - \frac{4}{s}\right)^{1/2} \int_0^1 dx \, \frac{(1-x)^m}{z_\ell x^2 + s(1-x)} \\ &= -2 \int_0^1 dx \, \frac{(1-x)^m}{x^2} \left[1 + \frac{\alpha(x)}{2} \log\left(\frac{\alpha(x) - 1}{\alpha(x) + 1}\right)\right] \end{aligned} \tag{129}$$

$$\alpha(x) = \left(1 + \frac{4(1-x)}{z_\ell \, x^2}\right)^{1/2} \tag{130}$$

We have checked that the integrals above are indeed convergent, although we could not find an analytic expression for $z_\ell \neq 1$.

A.3 Results

We are now ready to present results for the integrals appearing in $T_\ell^{e^2 p^4}$.

$$I_1^{(\ell)\pi\pi} = \frac{i}{(4\pi)^4} \left[-\frac{m_\pi^{4w}}{(4\pi)^{2w}} \frac{\Gamma(-w)\Gamma(-2w)\Gamma(1-w)}{(1+w)\Gamma(2-2w)} + \tilde{E}_0(z_\ell) - \tilde{E}_2(z_\ell) \right] \tag{131}$$

$$I_1^{(\pi)\pi\pi} = \frac{i}{(4\pi)^4} \left[-\frac{m_\pi^{4w}}{(4\pi)^{2w}} \frac{\Gamma(-w)\Gamma(-2w)\Gamma(1-w)}{(1+w)\Gamma(2-2w)} - \left(\frac{2}{3}\pi^2 - 7\right) \right] \tag{132}$$

$$\begin{aligned} I_2^{(\ell)\pi\pi} &= i \, m_\pi^{2+4w} \, [C(w)\Gamma(-w)]^2 \left[\frac{\Gamma(-1-w)\Gamma(-1-2w)}{\Gamma(-w)\Gamma(-2w)} \right. \\ &- \left. \frac{2z_\ell}{(1+w)(2+w)} \frac{\Gamma(-2w)\Gamma(1-w)}{\Gamma(-w)\Gamma(2-2w)} \right] \\ &+ i \frac{2\, m_\ell^2}{(4\pi)^4} \left[\tilde{E}_0(z_\ell) - \frac{1}{2}\tilde{E}_1(z_\ell) - \tilde{E}_2(z_\ell) + \frac{1}{2}\tilde{E}_3(z_\ell) \right] \end{aligned} \tag{133}$$

$$I_2^{(\pi)\pi\pi} = i \, m_\pi^{2+4w} \, [C(w)\Gamma(-w)]^2 \left[\frac{\Gamma(-1-w)}{\Gamma(-w)} + \frac{3}{2} - \frac{17}{4} w + \frac{59}{8} w^2 \right] \tag{134}$$

$$I_3^{\pi\pi} = i \, m_\pi^{2+4w} \, [C(w)\Gamma(-w)]^2 \, \frac{\Gamma(-1-w)\Gamma(-1-2w)}{\Gamma(-w)\Gamma(-2w)} \tag{135}$$

$$I_4^{\pi\pi} = i \, m_\pi^{4+4w} \, [C(w)\Gamma(-w)]^2 \left[\frac{\Gamma(-1-w)}{\Gamma(-w)} \right]^2 \tag{136}$$

$$\begin{aligned} I_5^{\pi\pi} &= I_4^{\pi\pi} + i4 \, m_\pi^{4+4w} \Big[z_\ell \frac{2+w}{4+2w} \, \tilde{E}(0, 1; z_\ell) \\ &- z_\ell^2 \left(\tilde{E}(0, 2; z_\ell) - 2\tilde{E}(1, 2; z_\ell) + \tilde{E}(2, 2; z_\ell) \right) \Big] \end{aligned} \tag{137}$$

$$T_1^{\pi\pi} = \frac{i}{(4\pi)^4}\frac{1/m_\pi^2}{z_\ell - 1}\int_4^\infty \frac{ds}{s}\left(1-\frac{4}{s}\right)^{1/2} t_1^{\pi\pi}(s) \equiv \frac{i}{(4\pi)^4\ m_\pi^2}\tilde{T}_1^{\pi\pi}(z_\ell) \tag{138}$$

$$t_1^{\pi\pi}(s) = \int_0^1 dx\,\frac{1}{x}\,\log\left(\frac{x^2\,z_\ell + s(1-x)}{x^2 + s(1-x)}\right)$$

$$T_2^{\pi\pi} = I_1^{(\ell)\pi\pi} + \frac{i}{(4\pi)^4}\int_4^\infty \frac{ds}{s}\left(1-\frac{4}{s}\right)^{1/2} t_2^{\pi\pi}(s) \equiv I_1^{(\ell)\pi\pi} + \frac{i}{(4\pi)^4}\tilde{T}_2^{\pi\pi}(z_\ell) \tag{139}$$

$$t_2^{\pi\pi}(s) = 1 + \frac{1}{z_\ell - 1}\int_0^1 dx\left(1+\frac{s}{x}-\frac{s}{x^2}\right)\log\left(\frac{x^2\,z_\ell + s(1-x)}{x^2+s(1-x)}\right)$$

Note that $t_{1,2}^{\pi\pi}(s)$ can be expressed in terms of elementary functions and Spence functions. The full expressions, however, are not particularly enlightening. Since $t_{1,2}^{\pi\pi}(s)$ are not singular for $z_\ell \to 0$, numerical integration is stable and sufficient for our purposes.

For the two-loop integrals involving $\bar{J}^{KK}(q^2)$ we find:

$$I_1^{(\ell)KK} = \frac{i}{(4\pi)^4}\left[-\frac{m_K^{4w}}{(4\pi)^{2w}}\frac{\Gamma(-w)\Gamma(-2w)\Gamma(1-w)}{(1+w)\Gamma(2-2w)} + \tilde{E}_0(\tilde{z}_\ell) - \tilde{E}_2(\tilde{z}_\ell)\right] \tag{140}$$

$$I_1^{(\pi)KK} = \frac{i}{(4\pi)^4}\left[-\frac{m_K^{4w}}{(4\pi)^{2w}}\frac{\Gamma(-w)\Gamma(-2w)\Gamma(1-w)}{(1+w)\Gamma(2-2w)} + \tilde{E}_0(\tilde{z}_\pi) - \tilde{E}_2(\tilde{z}_\pi)\right] \tag{141}$$

$$\begin{aligned} I_2^{(\ell)KK} &= i\,m_K^{2+4w}\,[C(w)\Gamma(-w)]^2\left[\frac{\Gamma(-1-w)\Gamma(-1-2w)}{\Gamma(-w)\Gamma(-2w)}\right.\\ &- \left.\frac{2\tilde{z}_\ell}{(1+w)(2+w)}\frac{\Gamma(-2w)\Gamma(1-w)}{\Gamma(-w)\Gamma(2-2w)}\right]\\ &+ i\frac{2\,m_\ell^2}{(4\pi)^4}\left[\tilde{E}_0(\tilde{z}_\ell) - \frac{1}{2}\tilde{E}_1(\tilde{z}_\ell) - \tilde{E}_2(\tilde{z}_\ell) + \frac{1}{2}\tilde{E}_3(\tilde{z}_\ell)\right] \end{aligned} \tag{142}$$

$$I_2^{(\pi)KK} = I_2^{(\ell)KK}\Big|_{m_\ell \to m_\pi} \tag{143}$$

$$I_3^{KK} = i\,m_K^{2+4w}\,[C(w)\Gamma(-w)]^2\,\frac{\Gamma(-1-w)\Gamma(-1-2w)}{\Gamma(-w)\Gamma(-2w)} \tag{144}$$

$$I_4^{KK} = i\,m_K^{4+4w}\,[C(w)\Gamma(-w)]^2\left[\frac{\Gamma(-1-w)}{\Gamma(-w)}\right]^2 \tag{145}$$

$$\begin{aligned} I_5^{KK} &= I_4^{KK} + i4\,m_K^{4+4w}\left[\tilde{z}_\ell\frac{2+w}{4+2w}\,\tilde{E}(0,1;\tilde{z}_\ell)\right.\\ &- \left.\tilde{z}_\ell^2\left(\tilde{E}(0,2;\tilde{z}_\ell) - 2\tilde{E}(1,2;\tilde{z}_\ell) + \tilde{E}(2,2;\tilde{z}_\ell)\right)\right] \end{aligned} \tag{146}$$

$$T_1^{KK} = \frac{i}{(4\pi)^4}\frac{1/m_K^2}{\tilde{z}_\ell - \tilde{z}_\pi}\int_4^\infty \frac{ds}{s}\left(1-\frac{4}{s}\right)^{1/2} t_1^{KK}(s) \equiv \frac{i}{(4\pi)^4\,m_K^2}\tilde{T}_1^{KK}(\tilde{z}_\ell,\tilde{z}_\pi) \tag{147}$$

$$t_1^{KK}(s) = \int_0^1 dx\,\frac{1}{x}\,\log\left(\frac{x^2\tilde{z}_\ell + s(1-x)}{x^2\tilde{z}_\pi + s(1-x)}\right)$$

$$
\begin{aligned}
T_2^{KK} &= I_1^{(\ell)KK} + \frac{i}{(4\pi)^4} \int_4^\infty \frac{ds}{s} \left(1 - \frac{4}{s}\right)^{1/2} t_2^{KK}(s) \\
&\equiv I_1^{(\ell)KK} + \frac{i}{(4\pi)^4} \tilde{T}_2^{KK}(\tilde{z}_\ell, \tilde{z}_\pi) \qquad (148) \\
t_2^{KK}(s) &= 1 + \frac{1}{\tilde{z}_\ell - \tilde{z}_\pi} \int_0^1 dx \left(\tilde{z}_\pi + \frac{s}{x} - \frac{s}{x^2}\right) \log\left(\frac{x^2 \tilde{z}_\ell + s(1-x)}{x^2 \tilde{z}_\pi + s(1-x)}\right)
\end{aligned}
$$

The finite loop contributions can all be expressed in terms logarithms and combinations of $\tilde{E}_n(x)$ and the following functions:

$$
\begin{aligned}
\tilde{R}_n(x) &= \frac{\tilde{E}_n(x)}{x} \qquad (149) \\
T^{\pi\pi}(x) &= \tilde{T}_2^{\pi\pi}(x) - 4\,\tilde{T}_1^{\pi\pi}(x) \qquad (150) \\
T^{KK}(x,y) &= \tilde{T}_2^{KK}(x,y) - 4\,\tilde{T}_1^{KK}(x,y)\ . \qquad (151)
\end{aligned}
$$

A.4 Standard form of two-loop integrals

A generic two-loop contribution can be cast in the following standard form ($C(w) = 1/(4\pi)^{2+w}$):

$$
\begin{aligned}
I^{2-\text{loops}} &= [C(w)\Gamma(-w)]^2\ m^{4w}\ x(d) \qquad (152) \\
x(d) &= x_0 + x_1\, w + x_2\, w^2 + O(w^3)\ , \qquad (153)
\end{aligned}
$$

with $m = m_\pi$ or $m = m_K$. Multiplying and dividing each contribution by $(\mu c)^{4w}$ [14], with

$$
\log c = -\frac{1}{2}\ (\log 4\pi - \gamma_E + 1) \qquad (154)
$$

and performing the expansion around $d = 4$, one finds:

$$
\begin{aligned}
I^{2-\text{loops}} &= \frac{(\mu c)^{4w}}{(4\pi)^4} \left[\frac{R^{(2)}}{w^2} + \frac{R^{(1)}}{w} + F + O(w)\right] \qquad (155) \\
R^{(2)} &= x_0 \qquad (156) \\
R^{(1)} &= x_1 + 2\,x_0 \left(\log\frac{m^2}{\mu^2} + 1\right) \qquad (157) \\
F &= x_2 + 2\,x_1 \left(\log\frac{m^2}{\mu^2} + 1\right) + x_0 \left[\frac{\pi^2}{6} + 2\left(\log\frac{m^2}{\mu^2} + 1\right)^2\right] \qquad (158)
\end{aligned}
$$

A.5 Standard form of one-loop integrals

The one loop diagrams with one insertion from the $O(p^4)$ effective lagrangian can be cast in a useful standard form as well. Denoting by $L(d)$ the generic d-dimensional p^4 LEC, one has:

$$I^{1-\text{loop}} = C(w)\Gamma(-w)\, m^{2w}\; L(d)\; y(d) \tag{159}$$

$$y(d) = y_0 + y_1\, w + y_2\, w^2 + O(w^3) \tag{160}$$

$$L(d) = \frac{(\mu c)^{2w}}{(4\pi)^2}\left(\frac{\Gamma}{2w} + (4\pi)^2 L^r(\mu)\right) \tag{161}$$

where $m = m_\pi$ or $m = m_K$ and the constant Γ determines the RG running of the renormalized coupling $L^r(\mu)$. Multiplying and dividing each contribution by $(\mu c)^{2w}$ [14], and performing the expansion around $d = 4$, one finds:

$$I^{1-\text{loop}} = \frac{(\mu c)^{4w}}{(4\pi)^4}\left[\frac{\tilde{R}^{(2)}}{w^2} + \frac{\tilde{R}^{(1)}}{w} + \tilde{F} + O(w)\right] \tag{162}$$

$$\tilde{R}^{(2)} = -\frac{\Gamma\, y_0}{2} \tag{163}$$

$$\tilde{R}^{(1)} = -(4\pi)^2\, L^r(\mu)\, y_0 - \frac{\Gamma\,(y_0 + y_1)}{2} - \frac{\Gamma\, y_0}{2}\log\frac{m^2}{\mu^2} \tag{164}$$

$$\begin{aligned}\tilde{F} &= -(4\pi)^2\, L^r(\mu)\,(y_0 + y_1) - \frac{\Gamma}{24}\left((6+\pi^2)y_0 + 12(y_1 + y_2)\right) \\ &- \left(2(4\pi)^2 L^r(\mu) y_0 + (y_0 + y_1)\Gamma\right)\log\frac{m}{\mu} - y_0\Gamma\left(\log\frac{m}{\mu}\right)^2\end{aligned} \tag{165}$$

The couplings of interest to us are L_9 and L_{10}, whose divergent parts are determined by:

$$\Gamma_9 = \frac{1}{4} \qquad \Gamma_{10} = -\frac{1}{4} \tag{166}$$

B Matching calculation

In this Appendix we report the details of our matching calculation. The intermediate steps of the calculation are:

1. Insert the large-N_C form factors of in the convolution representation of Eq. 22.

2. Reduce the resulting integrals to scalar Passarino-Veltman functions [34]. For these we follow the convention of Kniehl [35].

3. Expand the full result in powers of $m_{\ell,\pi}/M_V$, up to order $(m/M_V)^2$. This involves expanding the scalar integrals $B_0(p^2, m_1^2, m_2^2)$ and $C_0(...)$ in powers of ratios of the internal masses. This is trivial for B_0, somewhat less trivial for C_0. We derived a representation of C_0 as a two dimensional integral (see Ref. [36]) and used that as a starting point for the heavy mass expansion.

4. Subtract the ChPT$_\infty$ result from the expanded full result, thus obtaining the counterterm amplitude according to Eq. 69.

B.1 Reduction to Passarino Veltman functions

We use the conventions of Ref. [35] for the Passarino-Veltman functions, namely:

$$\{B_0, B_\mu, B_{\mu\nu}\}(p^2, m_1^2, m_2^2) = \int \frac{d^dq}{i\pi^2} \frac{\{1, q_\mu, q_\mu q_\nu\}}{[q^2 - m_1^2 + i\epsilon][(q+p)^2 - m_2^2 + i\epsilon]} \tag{167}$$

and

$$\{C_0, C_\mu, C_{\mu\nu}\}(p^2, k^2, (p+k)^2, m_1^2, m_2^2, m_3^2) = \\ -\int \frac{d^dq}{i\pi^2} \frac{\{1, q_\mu, q_\mu q_\nu\}}{[q^2 - m_1^2 + i\epsilon][(q+p)^2 - m_2^2 + i\epsilon][(q+p+k)^2 - m_3^2 + i\epsilon]}\ , \tag{168}$$

with

$$B_\mu = p_\mu\ B_1 \tag{169}$$

$$B_{\mu\nu} = p_\mu p_\nu\ B_{21} - g_{\mu\nu}\ B_{22} \tag{170}$$

$$C_\mu = p_\mu\ C_{11} + k_\mu\ C_{12} \tag{171}$$

$$C_{\mu\nu} = p_\mu p_\nu\ C_{21} + k_\mu k_\nu\ C_{22} + (p_\mu k_\nu + k_\mu p_\nu)\ C_{23} - g_{\mu\nu} C_{24}\ . \tag{172}$$

For the reduction of vector and tensor integrals to scalar Passarino-Veltman functions we have used the relations $(A.7)$, $(A.8)$ and $(A.9)$ of Ref. [35].

B.1.1 T_{V_1}

In the reduction of T_{V_1} we need the following tensor and vector integrals:

$$\int \frac{d^dq}{(2\pi)^d} \frac{V_1(q^2, W^2)}{q^2\,(q^2 - 2q\cdot p_\ell)}\, q^\alpha q^\beta = V_{\ell\ell}\, p_\ell^\alpha p_\ell^\beta + V_{\nu\nu}\, p_\nu^\alpha p_\nu^\beta + V_{\nu\ell}\,(p_\nu^\alpha p_\ell^\beta + p_\ell^\alpha p_\nu^\beta) + V_g\, g^{\alpha\beta} \tag{173}$$

$$\int \frac{d^dq}{(2\pi)^d} \frac{V_1(q^2, W^2)}{q^2\,(q^2 - 2q\cdot p_\ell)}\, q^\alpha = V_\pi\ p^\alpha + V_\ell\ p_\ell^\alpha \tag{174}$$

Using the above definitions the amplitude reads:

$$T_{V_1} = -ie^2 T_\ell^{p^2}\Big[6V_g + (m_\ell^2 - m_\pi^2)\Big(V_{\ell\ell} - V_{\nu\ell}\Big)\Big] \tag{175}$$

$$V_{\ell\ell} = \frac{i}{6(4\pi)^2}\left[\frac{1}{M_V^2}\big(B_{21}(m_\ell^2, M_V^2, m_\ell^2) - B_{21}(m_\ell^2, 0, m_\ell^2)\big) - (1-\kappa)\bar{C}_{21} - \kappa\bar{\bar{C}}_{21}\right] \tag{176}$$

$$V_{\nu\ell} = \frac{i}{6(4\pi)^2}\left[-(1-\kappa)\bar{C}_{23} - \kappa\bar{\bar{C}}_{23}\right] \tag{177}$$

$$V_g = \frac{i}{6(4\pi)^2}\left[-\frac{1}{M_V^2}\big(B_{22}(m_\ell^2, M_V^2, m_\ell^2) - B_{22}(m_\ell^2, 0, m_\ell^2)\big) + (1-\kappa)\bar{C}_{24} + \kappa\bar{\bar{C}}_{24}\right] \tag{178}$$

with

$$\kappa = \frac{2M_V^2 - m_\pi^2 - c_V}{M_V^2} \tag{179}$$

$$c_V = M_V^4 \frac{N_C}{4\pi^2 F^2} = -6\, M_V^4\, V_1 \tag{180}$$

$$\bar{C}_{ij} = C_{ij}(m_\ell^2, 0, m_\pi^2, 0, m_\ell^2, M_V^2) \tag{181}$$

$$\bar{\bar{C}}_{ij} = C_{ij}(m_\ell^2, 0, m_\pi^2, M_V^2, m_\ell^2, M_V^2)\ . \tag{182}$$

B.1.2 T_{A_1}

In the reduction of T_{A_1} we need the following tensor, vector, and scalar integrals:

$$\int \frac{d^dq}{(2\pi)^d} \frac{A_1(q^2, W^2)}{q^2\,(q^2 - 2q\cdot p_\ell)}\, q^\alpha q^\beta = A_{\ell\ell}\, p_\ell^\alpha p_\ell^\beta + A_{\nu\nu}\, p_\nu^\alpha p_\nu^\beta + A_{\nu\ell}(p_\nu^\alpha p_\ell^\beta + p_\ell^\alpha p_\nu^\beta) + A_g\, g^{\alpha\beta} \tag{183}$$

$$\int \frac{d^dq}{(2\pi)^d} \frac{A_1(q^2, W^2)}{q^2\,(q^2 - 2q\cdot p_\ell)}\, q^\alpha = A_\nu\, p_\nu^\alpha + A_\ell\, p_\ell^\alpha \tag{184}$$

$$\int \frac{d^dq}{(2\pi)^d} \frac{A_1(q^2, W^2)}{q^2 - 2q\cdot p_\ell} = -S_{A_1} \tag{185}$$

$$\int \frac{d^dq}{(2\pi)^d} \frac{A_1(q^2, W^2)}{q^2 - 2q\cdot p_\ell}\, q^\alpha = E_\nu\, p_\nu^\alpha + E_\ell\, p_\ell^\alpha \tag{186}$$

$$\int \frac{d^dq}{(2\pi)^d} \frac{A_1(q^2, W^2)}{q^2}\, q^\alpha = E_\pi\, p^\alpha \tag{187}$$

Using the above definitions the amplitude reads:

$$\begin{aligned} T_{A_1} &= -ie^2 T_\ell^{p^2} \Bigg\{ S_{A_1} - E_\pi + A_\nu (m_\pi^2 - m_\ell^2) \\ &+ (d-2)\left[A_g - E_\ell + \frac{m_\pi^2 + m_\ell^2}{2} A_{\ell\ell} + \frac{m_\pi^2 - m_\ell^2}{2} A_{\nu\ell} \right] \Bigg\} \end{aligned} \tag{188}$$

$$S_{A_1} = \frac{i}{(4\pi)^2} \left[b_1 \tilde{C}_0 + b_2\, B_0(0, m_\ell^2, M_A^2) + b_3\, B_0(m_\ell^2, M_V^2, m_\ell^2) \right] \tag{189}$$

$$A_\nu = \frac{i}{(4\pi)^2} \left[\frac{b_1}{M_V^2} \tilde{C}_{12} - \left(\frac{b_1}{M_V^2} + b_2 \right) \tilde{\tilde{C}}_{12} \right] \tag{190}$$

$$\begin{aligned} A_{\ell\ell} &= \frac{-i}{(4\pi)^2} \Bigg[\frac{b_3}{M_V^2} \left(B_{21}(m_\ell^2, M_V^2, m_\ell^2) - B_{21}(m_\ell^2, 0, m_\ell^2) \right) \\ &+ \frac{b_1}{M_V^2} \tilde{C}_{21} - \left(\frac{b_1}{M_V^2} + b_2 \right) \tilde{\tilde{C}}_{21} \Bigg] \end{aligned} \tag{191}$$

$$A_{\nu\ell} = \frac{-i}{(4\pi)^2} \left[\frac{b_1}{M_V^2} \tilde{C}_{23} - \left(\frac{b_1}{M_V^2} + b_2 \right) \tilde{\tilde{C}}_{23} \right] \tag{192}$$

$$
\begin{aligned}
A_g &= \frac{-i}{(4\pi)^2}\left[-\frac{b_3}{M_V^2}\left(B_{22}(m_\ell^2, M_V^2, m_\ell^2) - B_{22}(m_\ell^2, 0, m_\ell^2)\right)\right. \\
&- \left.\frac{b_1}{M_V^2}\tilde{C}_{24} + \left(\frac{b_1}{M_V^2} + b_2\right)\tilde{\tilde{C}}_{24}\right] \qquad (193) \\
E_\ell &= \frac{i}{(4\pi)^2}\left[b_1\tilde{C}_{11} - b_2 B_0(0, m_\ell^2, M_A^2) + b_3 B_1(m_\ell^2, M_V^2, m_\ell^2)\right] \qquad (194) \\
E_\pi &= \frac{i}{(4\pi)^2}\left[-\frac{b_1}{M_V^2}B_1(m_\pi^2, M_V^2, M_A^2) + \left(\frac{b_1}{M_V^2} + b_2\right)B_1(m_\pi^2, 0, M_A^2)\right] \ , \qquad (195)
\end{aligned}
$$

with

$$
\begin{aligned}
b_1 &= M_V^2(1-b_2) - M_A^2(1+b_3) \qquad (196) \\
\tilde{C}_{ij} &= C_{ij}(m_\ell^2, 0, m_\pi^2, M_V^2, m_\ell^2, M_A^2) \qquad (197) \\
\tilde{\tilde{C}}_{ij} &= C_{ij}(m_\ell^2, 0, m_\pi^2, 0, m_\ell^2, M_A^2) \ . \qquad (198)
\end{aligned}
$$

B.1.3 T_{A_2}

In the reduction of T_{A_2} we need the following vector and scalar integrals:

$$
\begin{aligned}
\int \frac{d^d q}{(2\pi)^d}\frac{A_2(q^2, W^2)}{q^2 - 2q\cdot p_\ell} &= S_{A_2} \qquad (199) \\
\int \frac{d^d q}{(2\pi)^d}\frac{A_2(q^2, W^2)}{q^2 - 2q\cdot p_\ell}\, q^\alpha &= F_\nu\, p_\nu^\alpha + F_\ell\, p_\ell^\alpha \qquad (200) \\
\int \frac{d^d q}{(2\pi)^d}\frac{A_2(q^2, W^2)}{q^2}\, q^\alpha &= \tilde{F}_\pi\, p^\alpha \qquad (201)
\end{aligned}
$$

Using the above definitions the amplitude reads:

$$
\begin{aligned}
T_{A_2} &= ie^2 T_\ell^{p^2}\left[-2S_{A_2} + (2-d)F_\ell - \tilde{F}_\pi\right] \qquad (202) \\
S_{A_2} &= \frac{i}{(4\pi)^2}\left[(2+d_2)M_A^2\tilde{C}_0 - d_2\, B_0(m_\ell^2, M_V^2, m_\ell^2)\right] \qquad (203) \\
F_\ell &= -\frac{i}{(4\pi)^2}\left[(2+d_2)M_A^2\tilde{C}_{11} - d_2 B_1(m_\ell^2, M_V^2, m_\ell^2)\right] \qquad (204) \\
\tilde{F}_\pi &= \frac{i}{(4\pi)^2}\frac{M_A^2}{M_V^2}(2+d_2)\left[B_1(m_\pi^2, M_V^2, M_A^2) - B_1(m_\pi^2, 0, M_A^2)\right] \ . \qquad (205)
\end{aligned}
$$

B.1.4 T_{F_V}

The F_V-induced amplitude reads:

$$
\begin{aligned}
T_{F_V} &= 2\frac{e^2}{(4\pi)^2}T_\ell^{p^2}\left\{(m_\pi^2 + m_\ell^2)\, C_0(m_\ell^2, 0, m_\pi^2, M_V^2, m_\ell^2, m_\pi^2)\right. \\
&+ \left.\frac{1}{m_\pi^2 - m_\ell^2}\left[m_\ell^2\, B_0(m_\pi^2, M_V^2, m_\pi^2) - m_\pi^2\, B_0(m_\ell^2, M_V^2, m_\ell^2)\right]\right\} . \qquad (206)
\end{aligned}
$$

B.2 Expansion of the relevant three-point scalar functions

We use the following representation for the C_0 function as a basis for the large mass expansion:

$$C_0(p^2,k^2,(p+k)^2,m_1^2,m_2^2,m_3^2) = \int_0^1 dx \int_0^{1-x} dy \; \frac{1}{ax^2+by^2+cxy+dx+ey+f} \; , \quad (207)$$

with

$$\begin{aligned}
a &= (p+k)^2 \\
b &= p^2 \\
c &= (p+k)^2 + p^2 - k^2 \\
d &= m_3^2 - m_1^2 - (p+k)^2 \\
e &= m_2^2 - m_1^2 - p^2 \\
f &= m_1^2 \; .
\end{aligned}$$

We then find (we give results up to the needed order):

$$C_0(m_\ell^2,0,m_\pi^2,M_V^2,m_\ell^2,m_\pi^2) = \frac{1}{m_\pi^2-m_\ell^2}\frac{1}{M_V^2}\left(m_\pi^2\,\log\frac{M_V^2}{m_\pi^2} - m_\ell^2\,\log\frac{M_V^2}{m_\ell^2}\right) + \ldots \quad (208)$$

$$\begin{aligned}
C_0(m_\ell^2,0,m_\pi^2,0,m_\ell^2,M_V^2) &= \frac{1}{M_V^2}\left(1+\log\frac{M_V^2}{m_\ell^2}\right) + \frac{m_\pi^2+m_\ell^2}{4\,M_V^4}\left(1+2\log\frac{M_V^2}{m_\ell^2}\right) \\
&+ \frac{m_\pi^4+m_\ell^2m_\pi^2+m_\ell^4}{9\,M_V^6}\left(1+3\log\frac{M_V^2}{m_\ell^2}\right) \\
&+ \frac{m_\pi^6+m_\ell^2m_\pi^4+m_\ell^4m_\pi^2+m_\ell^6}{16\,M_V^8}\left(1+4\log\frac{M_V^2}{m_\ell^2}\right) + \ldots \quad (209)
\end{aligned}$$

$$\begin{aligned}
C_0(m_\ell^2,0,m_\pi^2,M_V^2,m_\ell^2,M_V^2) &= \frac{1}{M_V^2} + \frac{1}{M_V^4}\left(\frac{5}{4}m_\ell^2 + \frac{1}{12}m_\pi^2 - m_\ell^2\log\frac{M_V^2}{m_\ell^2}\right) \\
&+ \frac{1}{M_V^6}\left(\frac{28}{9}m_\ell^4 - \frac{5}{36}m_\pi^2m_\ell^2 + \frac{1}{90}m_\pi^4 - 3m_\ell^4\log\frac{M_V^2}{m_\ell^2}\right) + \ldots \quad (210)
\end{aligned}$$

$$\begin{aligned}
C_0(m_\ell^2,0,m_\pi^2,M_V^2,m_\ell^2,M_A^2) &= \frac{1}{M_V^2}\frac{1}{z_A^2-1}\log z_A^2 + \frac{1}{M_V^4}f^{(4)}\left(z_A^2,m_\ell^2,m_\pi^2\right) \\
&+ \frac{1}{M_V^6}f^{(6)}\left(z_A^2,m_\ell^2,m_\pi^2\right) + \ldots \; , \quad (211)
\end{aligned}$$

with $z_A = M_A/M_V$. The functions $f^{(4,6)}(z_A^2, m_\ell^2, m_\pi^2)$ have a simple but lengthy expression:

$$\begin{aligned} f^{(4)}(z_A^2, m_\ell^2, m_\pi^2) &= -\frac{m_\ell^2}{z_A^2}\log\frac{M_V^2}{m_\ell^2} + \frac{(-2m_\pi^2 + m_\ell^2(-1+z_A^2))}{2(-1+z_A^2)^2} \\ &+ \frac{(m_\pi^2 z_A^2(1+z_A^2) + m_\ell^2(2-3z_A^2+z_A^4))}{z_A^2(-1+z_A^2)^3}\log z_A \qquad (212) \\ f^{(6)}(z_A^2, m_\ell^2, m_\pi^2) &= -\frac{m_\ell^4(1+2z_A^2)}{z_A^4}\log\frac{M_V^2}{m_\ell^2} + \frac{m_\ell^4(3-3z_A^2+z_A^4)}{3z_A^4(-1+z_A^2)^3}\log z_A^2 \\ &+ \frac{m_\ell^2 m_\pi^2(4-5z_A^2+z_A^4) + m_\pi^4(1+4z_A^2+z_A^4)}{3(-1+z_A^2)^5}\log z_A^2 \\ &+ \frac{m_\ell^4(3-15z_A^2+10z_A^4)}{6z_A^2(-1+z_A^2)^2} - \frac{m_\pi^4(1+z_A^2)}{(-1+z_A^2)^4} \\ &- \frac{m_\ell^2 m_\pi^2(3+2z_A^2-7z_A^4+2z_A^6)}{6z_A^2(-1+z_A^2)^4} \ . \qquad (213) \end{aligned}$$

B.3 Results

Recalling the definition $z_A = M_A/M_V$ and neglecting as usual the m_ℓ-independent terms that drop in $R_{e/\mu}$, we find:

$$T_{F_V}^{CT} = T_\ell^{p^2}\,\frac{\alpha}{4\pi}\frac{m_\ell^2}{M_V^2}\,2\,\log\frac{M_V^2}{\mu^2} \qquad (214)$$

$$T_{V_1}^{CT} = T_\ell^{p^2}\,\frac{\alpha}{4\pi}\frac{m_\ell^2}{M_V^2}\left[-\frac{4}{9} - \frac{19}{9}V_1\,M_V^2 + \frac{4}{3}V_1\,M_V^2\,\log\frac{M_V^2}{\mu^2}\right] \qquad (215)$$

$$\begin{aligned} T_{A_1}^{CT} &= T_\ell^{p^2}\,\frac{\alpha}{4\pi}\frac{m_\ell^2}{M_V^2}\left[\frac{7}{3}\left(1-\frac{1}{z_A^2}\right)\log\frac{M_V^2}{\mu^2}\right. \\ &- \left.\frac{37-63z_A^2+21z_A^4+5z_A^6+12(7-10z_A^2+4z_A^4)\log z_A}{18z_A^2(-1+z_A^2)^2}\right] \qquad (216) \end{aligned}$$

$$\begin{aligned} T_{A_2}^{CT} &= T_\ell^{p^2}\,\frac{\alpha}{4\pi}\frac{m_\ell^2}{M_V^2}\left[-8\log\frac{M_V^2}{\mu^2} + \frac{12-16z_A^2+4z_A^6+4(12-15z_A^2+5z_A^4)\log z_A}{3(-1+z_A^2)^3}\right. \\ &+ \left. d_2\,\frac{8-14z_A^2+6z_A^4+2(12-15z_A^2+5z_A^4)\log z_A}{3(-1+z_A^2)^3}\right] \ . \qquad (217) \end{aligned}$$

Using the input from Ref. [27] the second line of $T_{A_1}^{CT}$ should be replaced by:

$$\frac{37-148z_A^2+168z_A^4-52z_A^6-5z_A^8+12(7-28z_A^2+27z_A^4-8z_A^6)\log z_A}{18z_A^2(-1+z_A^2)^3} \qquad (218)$$

By comparing these expressions with the ChPT ones, one can easily verify that the matching procedure captures in full the single-log renormalization scale dependence, as expected from the $1/N_C$ expansion.

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